Math 157 Exam 3

Monday, February 11, 2002

1. Evaluate the following integrals

(a) $\displaystyle\int \sec x \, \tan x \, dx$ (c) $\displaystyle\int \frac{dx}{x+1}$

(b) $\displaystyle\int_0^\infty e^{-x}\, dx$ (d) $\displaystyle\int \cos^2 x \, dx$

2. Each of the following expressions defines a function of x. Compute its derivative with respect to x, in the simplest possible form.

(a) $\displaystyle\frac{x}{1+e^{1/x}}$ (c) $\displaystyle\int_x^{2x} \frac{dt}{1+t^2}$

(b) $\displaystyle\int_3^x \left(5^t + t^5\right) dt$ (d) $\displaystyle\arcsin \frac{x-1}{x+1}.$

(In (a), evaluate the left and right derivatives separately at $x = 0$.)

3. For each of the following functions, draw a careful graph, indicating whether the function is odd, even, or periodic, how it behaves at ∞, and where it has singularities.

(a) $\displaystyle x \arctan \frac{1}{x}$ (c) $e^{-1/x}$

(b) $\log\bigl(\cos(x)\bigr)$ (d) $\displaystyle\arcsin\left(\frac{1}{2}\sin x\right)$

4. In each of the following, f is a continuous function on $[0, 1]$.

   (a) Show that
$$\int_0^\pi f(\sin x)\, \cos x \, dx = 0.$$

   (b) Characterize the set of f having the property that
$$\int_0^x f(t)\, dt \;=\; \int_x^1 f(t)\, dt \qquad \text{for all } x \in [0, 1].$$

5. (a) Show that for any positive integer n and any $x > 0$,
$$n\left(1 - \frac{1}{x^{1/n}}\right) \;\le\; \log x \;\le\; n\left(x^{1/n} - 1\right).$$

   (*Hint:* Compare the functions t^{-1}, $t^{-1+(1/n)}$, and $t^{-1-(1/n)}$.

   (b) By substituting $x = e^y$, show that
$$\left(1 + \frac{y}{n}\right)^n \;\le\; e^y \;\le\; \left(1 - \frac{y}{n}\right)^{-n} \qquad \text{for } y < n.$$