Math 157 Analysis I — Solution of Term Exam 2

web version:
http://www.math.toronto.edu/~drorbn/classes/0203/157AnalysisI/TermExam2/Solution.html

Problem 1. Prove that there is a real number x so that

$$x^{157} + \frac{157}{1 + x^2 + \cos^2 x} = 157.$$

If your proof uses the intermediate value theorem, state it clearly and prove that it follows from the postulate P13.

Solution. As a composition/sum/quotient of continuous functions, the left hand side is a continuous function of x. The term $\frac{157}{1+x^2+\cos^2 x}$ is bounded by 157 and hence the large x behaviour of the left hand side is dominated by that of x^{157}. Thus for large negative x the left hand side goes to $-\infty$ and for large positive x it goes to $+\infty$. Thus by the intermediate value theorem the left hand side must attain the value 157 for some x.

Our proof does use the intermediate value theorem, and hence its statement and proof should be reproduced. See Spivak's chapter 8.

Problem 2.

1. Define in precise terms "f is differentiable at a".
2. Let

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Is f differentiable at 0? If you think it is, prove your assertion and compute $f'(0)$. Otherwise prove that it isn't.

Solution.

1. A function f is said to be differentiable at a point a if the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists.

2. According to the definition of differentiability, we consider the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(h)}{h}.$$

We claim that this limit is 0 and hence $f'(0)$ exists and is equal to 0. Indeed, Let $\epsilon > 0$ be any positive number and set $\delta = \epsilon$. Now if h satisfies $0 < |h| < \delta$ is rational then $\left|\frac{f(h)}{h} - 0\right| = \left|\frac{h^2}{h}\right| = |h| < \delta = \epsilon$ and if h satisfies $0 < |h| < \delta$ is irrational then $\left|\frac{f(h)}{h} - 0\right| = \left|\frac{0}{h}\right| = 0 < \epsilon$, so in general $0 < |h| < \delta$ implies $\left|\frac{f(h)}{h} - 0\right| < \epsilon$. Thus $\lim_{h\to 0} \frac{f(h)}{h} = 0$ as asserted above.

Problem 3. Calculate dy/dx in each of the following cases. Your answer may be in terms of x, of y, or of both, but reduce it algebraically to a reasonably simple form. You do not need to specify the domain of definition.

(a) $x^3 + y^3 = 2$

(b) $y = x/\sqrt{x^2 - 4}$

(c) $y^4 + y^3 + xy = 1$

(d) $y = \sin\bigl(\sin(x)\bigr)$

Solution.

(a) Differentiating both sides with respect to x we get $3x^2 + 3y^2y' = 0$ and hence $\frac{dy}{dx} = y' = -\frac{x^2}{y^2}$.

(b) Using the rule for differentiating a quotient, then the chain rule and then simplifying a bit, we get

$$\frac{dy}{dx} = \frac{x'\sqrt{x^2-4} - x\left(\sqrt{x^2-4}\right)'}{\left(\sqrt{x^2-4}\right)^2} = \frac{\sqrt{x^2-4} - \frac{1}{2}x\,(x^2-4)^{-1/2}\cdot 2x}{x^2-4} = -\frac{4}{(x^2-4)^{3/2}}.$$

(c) Differentiating both sides with respect to x we get $4y^3y' + 3y^2y' + y + xy' = 0$ and hence $y' = \dfrac{-y}{x + 3y^2 + 4y^3}$.

(d) Using the chain rule, $y' = \cos(\sin(x))\cos(x)$.

Problem 4.

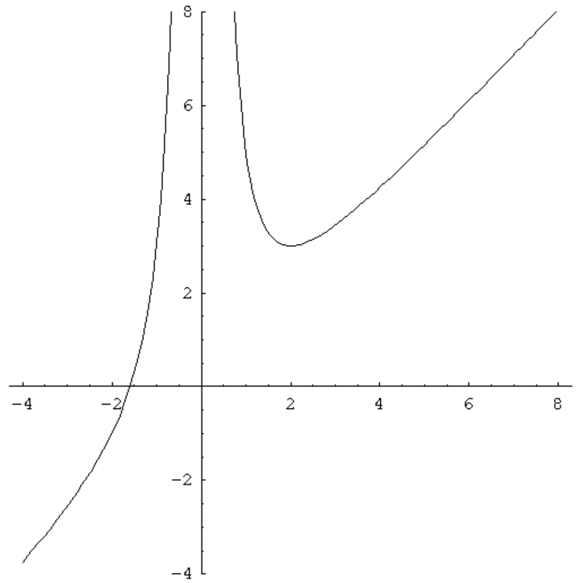
1. Prove that if $f'(x) > 0$ on some interval then f is increasing on that interval.

2. Sketch the graph of the function $f(x) = x + \dfrac{4}{x^2}$.

Solution.

1. See Spivak chapter 11.

2. $f(0)$ is not defined; $\lim_{x\to 0} f(x) = +\infty$. The only solution to $f(x) = 0$ is $x = -\sqrt[3]{4}$, so the point $(-\sqrt[3]{4}, 0)$ is on the graph. $f'(x) = 1 - 8/x^3$; this is positive when $x > \sqrt[3]{8} = 2$ and when $x < 0$ and negative when $0 < x < 2$, so f is increasing when $x > 2$ and when $x < 0$ and decreasing when $0 < x < 2$. The derivative is 0 only at $x = 2$; right before, the function is decreasing and right after it is increasing. So $x = 2$ is a local max and we can compute $f(2) = 3$. Finally, $\lim_{x\to\pm\infty} = \pm\infty$ and near ∞ our graph $y = f(x)$ is very close to $y = x$, so we arrive at the following graph:



Problem 5. Write a formula for $(f^{-1})''(x)$ in terms of f', f'' and $f^{-1}(x)$. Under what conditions does your formula hold?
Solution. From class material we knot that if f is continuous and $1-1$ near $f^{-1}(x)$, differentiable at $f^{-1}(x)$, and $f'(f^{-1}(x)) \neq 0$ then $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$. Using this we get

$$(f^{-1})''(x) = \left(\frac{1}{f'(f^{-1}(x))}\right)' = -\frac{(f'(f^{-1}(x)))'}{(f'(f^{-1}(x)))^2}$$
$$= -\frac{f''(f^{-1}(x)) \cdot (f^{-1})'(x)}{(f'(f^{-1}(x)))^2} = -\frac{f''(f^{-1}(x)) \cdot \frac{1}{f'(f^{-1}(x))}}{(f'(f^{-1}(x)))^2} = -\frac{f''(f^{-1}(x))}{(f'(f^{-1}(x)))^3}.$$

In the last chain of equalities we've used the chain rule, for which, in addition to what we already have, we need to know that f' is continuous around $f^{-1}(x)$ and differentiable at $f^{-1}(x)$ and the rule for differentiating a quotient, for which we need nothing new. Hence the full list of conditions needed for aour formula to hold is:

- f is $1-1$ near $f^{-1}(x)$.
- f is differentiable around $f^{-1}(x)$.
- $f'(f^{-1}(x)) \neq 0$.
- f is twice differentiable at $f^{-1}(x)$.

The results. 86 students took the exam; the average grade is 70.76, the median is 72 and the standard deviation is 18.35.