University of Toronto

Department of Mathematics

Math 157 Exam 2 Makeup

Week of December 17–21, 2001

One hour and fifty minutes

There are five problems, each worth 20 points although they do not have equal difficulty. Write your answer in the space below the problem; use the back of the sheets and the last page for scratch paper. Only work appearing on the front of the page will be graded. Write your name on each page.

No calculators

1	/20
2	/20
3	/20
4	/20
5	/20
Total	/100

1. Calculate dy/dx in each of the following cases. Your answer may be in terms of x, of y, or of both, but reduce it algebraically to a reasonably simple form. You do not need to specify the domain of definition.

   (a) $x^3 + y^3 = 2$

   (b) $y = x/\sqrt{x^2 - 4}$

   (c) $(xy)^2 = 1$

   (d) $y = \dfrac{t}{1+t}, \ y = 1 + t^2$

   (e) $x = \sin\bigl(\sin(x)\bigr)$

2. Graph the function

$$f(x) \;=\; \frac{x^3}{x^2+1},$$

showing clearly its behavior at $\pm\infty$, points where it is not defined, and ranges of x on which it is increasing, decreasing, convex, or concave. Find all local maxima, minima, and critical points, but you do not need to calculate precisely the points of inflection.

3. Evaluate the following limits if they exist

(a) $\lim_{x \to 0} \dfrac{\cos x - 1}{\sin x - x}$

(b) $\lim_{x \to \infty} x \sin \dfrac{1}{x}$

(c) $\lim_{x \to 0} \dfrac{\sin x}{\cos x}$

(d) $\lim_{h \to 0} \dfrac{\sqrt{4+h} - 2}{h}$

4. f is continuous on $a \leq x \leq b$, and has a second derivative $f''(x)$ at each point $a < x < b$. The straight line joining $(a, f(a))$ to $(b, f(b))$ intersects the graph of f in at least one point c with $a < c < b$. Prove that there is at least one point t with $a < t < b$ at which $f''(t) = 0$.

5. Define $f(x) = \frac{1}{3}x^3 + tx$, where t may take any real value. Let $m(t)$ be the minimum value of $f(x)$ on the interval $0 \le x \le 1$. ($m(t)$ is the minimum *value*; it does not depend on *where* the minimum is taken).

   (a) On one graph, plot $f(x)$ for $t = 0$, for $t = \pm 1$, and for $t = \pm 2$.

   (b) Does $m(t)$ exist for all values of t?

   (c) Plot $m(t)$ as a function of t, wherever it exists.

   (d) How is the graph of $f(x)$ (on $0 \le x \le 1$) for $t = t_1$ related to the graph of $f(x)$ for $t = t_2$, if $t_1 < t_2$? Prove that $m(t)$ is nondecreasing.