Math 157 Analysis I — Term Exam 1

University of Toronto, October 21, 2002.

Name: _____   Student ID: _____

Solve the following 5 problems.   Each is worth 20 points although they may have unequal difficulty. Write your answers in the space below the problems and on the front sides of the extra pages; use the back of the pages for scratch paper. Only work appearing on the front side of pages will be graded. Write your name and student number on each page. If you need more paper please ask the tutors. You have an hour and 50 minutes.

Allowed Material: Any calculating device that is not capable of displaying text.

Good Luck!

For Grading Use Only

1	/20
2	/20
3	/20
4	/20
5	/20
Total	/100

Problem 1.

1. Prove directly from the postulates for the real numbers and from the relevant definitions that if $a, b \geq 0$ and $a^2 < b^2$, then $a < b$. If you plan to use a formula such as $b^2 - a^2 = (b - a)(b + a)$ you don't need to prove it, but of course you have to be very clear about how it is used.

2. Use induction to prove that any integer n can be written in exactly one of the following two forms: $n = 2k$ or $n = 2k + 1$, where k is also an integer.

3. Prove that there is no rational number r such that $r^3 = 2$.

Problem 2.

1. Suppose $f(x) = x + 1$. Are there any functions g such that $f \circ g = g \circ f$?

2. Suppose that f is a constant function. For which functions g does $f \circ g = g \circ f$?

3. Suppose that $f \circ g = g \circ f$ for *all* functions f. Show that g is the identity function $g(x) = x$.

Problem 3. Sketch, to the best of your understanding, the graph of the function

$$f(x) = x^2 - \frac{1}{x^2}.$$

(What happens for x near 0? For large x? Where does the graph lie relative to the graph of the function $y = x^2$?)

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Problem 4.   Write the definition of $\lim_{x \to a} f(x) = l$ and give examples to show that the following definitions of $\lim_{x \to a} f(x) = l$ do not agree with the standard one:

1. For all $\delta > 0$ there is an $\epsilon > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - l| < \epsilon$.

2. For all $\epsilon > 0$ there is a $\delta > 0$ such that if $|f(x) - l| < \epsilon$, then $0 < |x - a| < \delta$.

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Problem 5.   Suppose that g is continuous at 0 and $g(0) = 0$ and that $|f(x)| \leq \sqrt{|g(x)|}$ for all x. Show that f is continuous at 0.