Problem 2

Solution:

(ii) $$\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^4} = \lim_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2}}{3x^3} = \lim_{x\to 0}\frac{-\sin x + x}{9x^2} = \lim_{x\to 0}\frac{-\cos x + 1}{18x}$$

$$= \lim_{x\to 0}\frac{\sin x}{18} = 0$$

(iv) $$\lim_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \lim_{x\to 0}\frac{-\sin x + x}{3x^3} = \lim_{x\to 0}\frac{-\cos x + 1}{9x^2} = \lim_{x\to 0}\frac{\sin x}{18x} = \lim_{x\to 0}\frac{\cos x}{18} = \frac{1}{18}$$

(vi) $$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0}\frac{\sin x - x}{x\sin x} = \lim_{x\to 0}\frac{\cos x - 1}{x\cos x + \sin x} = \lim_{x\to 0}\frac{-\sin x}{-x\sin x + \cos x + \sin x} = 0$$

Problem 4, ***see the last page***.

Problem 9

Proof:

(a) $\because x + y \neq kp + \frac{p}{2} \Leftrightarrow \cos(x+y) \neq 0$

$$\therefore \quad \tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x\cos y + \cos x\sin y}{\sin x\sin y - \cos x\cos y} = \frac{\frac{\sin x\cos y}{\sin x\sin y} + \frac{\cos x\sin y}{\sin x\sin y}}{\frac{\sin x\sin y}{\sin x\sin y} - \frac{\cos x\cos y}{\sin x\sin y}}$$

$$= \frac{\tan x + \tan y}{1 - \tan x\tan y}$$

(b) Let $a = \arctan x, b = \arctan y$, from (a) above: $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b}$

$$\Leftrightarrow \arctan(\tan(a+b)) = \arctan\frac{\tan a + \tan b}{1 - \tan a\tan b}$$

$$\Leftrightarrow a + b = \arctan\frac{\tan a + \tan b}{1 - \tan a\tan b}$$

noticed that $\tan a = \tan\arctan x = x, \tan b = \tan\arctan y = y$

$above \Leftrightarrow a + b = \arctan\frac{x+y}{1-xy}$, that is $\arctan x + \arctan y = \arctan\frac{x+y}{1-xy}$

Problem 15

(a) $\because \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$$\therefore \cos^2 x = \frac{\cos 2x + 1}{2}, \sin^2 x = \frac{1 - \cos 2x}{2}$$

(b) From (a): $\left(\cos\frac{x}{2}\right)^2 = \frac{\cos 2\cdot\frac{x}{2}+1}{2} = \frac{1+\cos x}{2},\ \left(\sin\frac{x}{2}\right)^2 = \frac{1-\cos 2\cdot\frac{x}{2}}{2} = \frac{1-\cos x}{2}$

$$0 \le x \le \frac{p}{2} \Rightarrow \cos\frac{x}{2} \ge 0,\ \sin\frac{x}{2} \ge 0$$

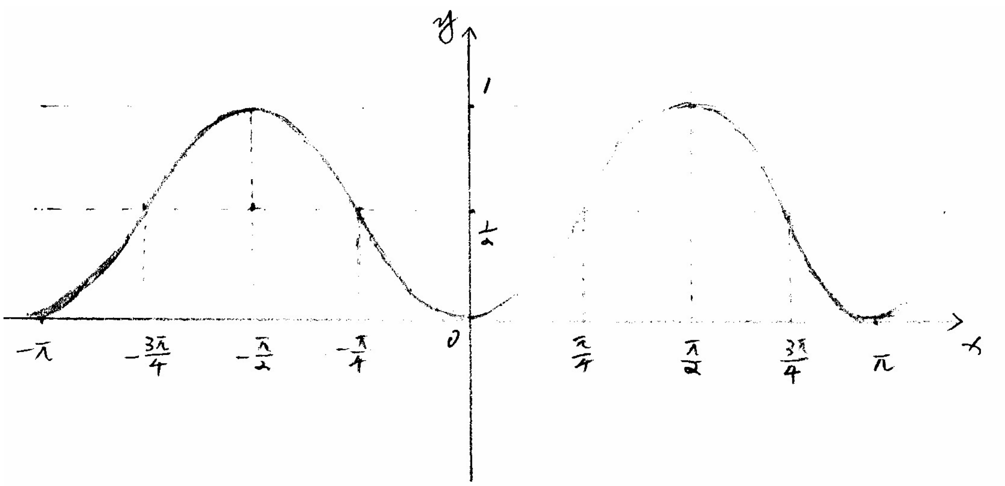
$$\Leftrightarrow \cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}},\ \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$$

(c) $\because \cos^2 x = \frac{\cos 2x+1}{2},\ \sin^2 x = \frac{1-\cos 2x}{2}$

$$\therefore \int_a^b \cos^2 x\,dx = \frac{1}{2}\left(\int_a^b \cos 2x\,dx + b - a\right) = \frac{1}{2}b - \frac{1}{2}a + \frac{1}{4}\sin 2b - \frac{1}{4}\sin 2a,$$

$$\int_a^b \sin^2 x\,dx = \frac{1}{2}\left(b - a - \int_a^b \cos 2x\,dx\right) = \frac{1}{2}b - \frac{1}{2}a - \frac{1}{4}\sin 2b + \frac{1}{4}\sin 2a$$

(d)



Problem 18

(a) Proof:

$$\sin\left(x+\frac{p}{2}\right) = \sin x\cos\frac{p}{2} + \cos x\sin\frac{p}{2} = \sin x\cdot 0 + \cos x\cdot 1 = \cos x$$

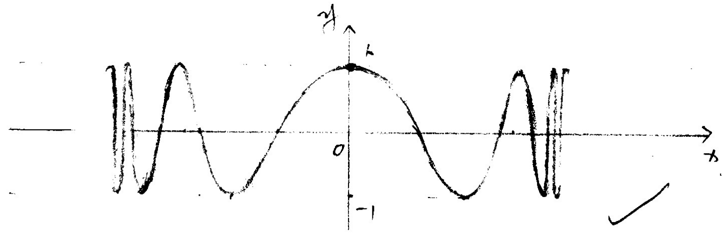
(b) Solution:

$$\arcsin(\cos x) = \arcsin\left(\sin\left(x+\frac{p}{2}\right)\right) = \begin{cases} x+\frac{p}{2}+2kp & if\ \exists\, k\in Z,\ x+\frac{p}{2}+2kp \in \left[-\frac{p}{2},\frac{p}{2}\right] \\ \frac{p}{2}-x+2kp & if\ \exists\, k\in Z,\ \frac{p}{2}-x+2kp \in \left[-\frac{p}{2},\frac{p}{2}\right] \end{cases}$$

$$\arccos(\sin x) = \arccos\left(\cos\left(x-\frac{p}{2}\right)\right) = \begin{cases} x-\frac{p}{2}+2kp & if\ \exists\, k\in Z,\ x-\frac{p}{2}+2kp \in [0,p] \\ \frac{p}{2}-x+2kp & if\ \exists\, k\in Z,\ \frac{p}{2}-x+2kp \in [0,p] \end{cases}$$

4. Graphs

(b) $f(x) = \sin(x^2)$



(d) $f(x) = \tan x - x$

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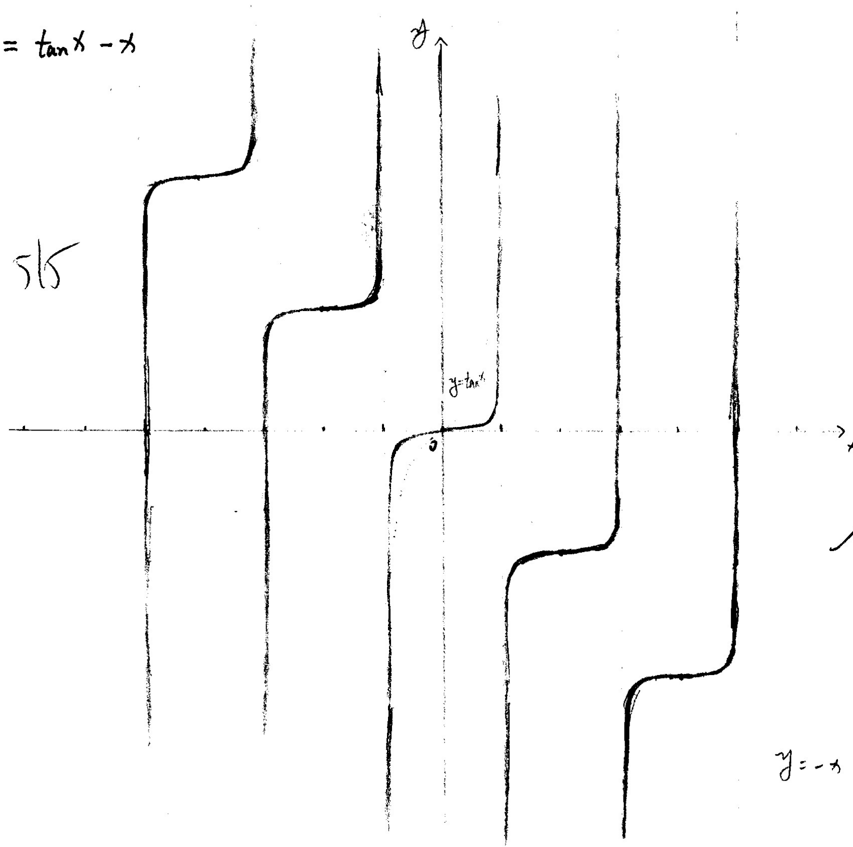
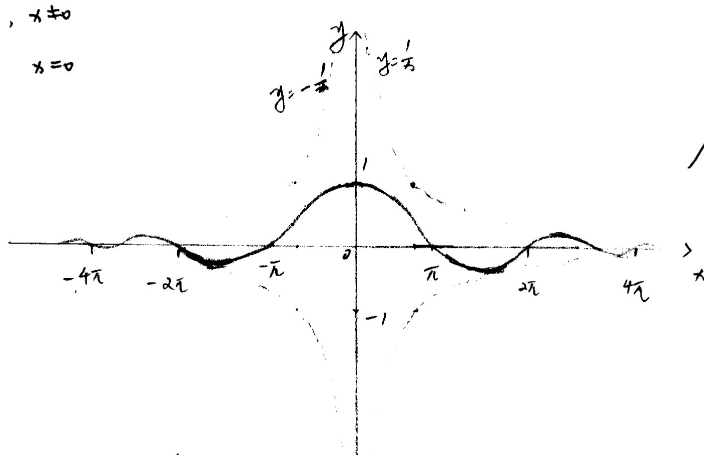


(f) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$



(period = 2π, fixed)