Chapter 9

1, (a) Proof:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\frac{1}{a+h}-\frac{1}{a}}{h}=\lim_{h\to 0}\frac{\frac{a-(a+h)}{(a+h)\cdot a}}{h}=\lim_{h\to 0}\frac{-1}{(a+h)\cdot a}=-\frac{1}{a^2}$$

(b) Proof:

From (a) we know the slope of tangent line at (a,1/a) is $-\frac{1}{a^2}$, then the equation of tangent line can be written as: $-\frac{1}{a^2}=\frac{y-\frac{1}{a}}{x-a}\,(x\neq a)\Leftrightarrow y=-\frac{1}{a^2}x+\frac{2}{a}$, the graph of $y=-\frac{1}{a^2}x+\frac{2}{a}$ and $f(x)=\frac{1}{x}$ intersect at point $\left(x,-\frac{1}{a^2}x+\frac{2}{a}\right)=\left(x,\frac{1}{x}\right)$, thus:

$$\frac{1}{x}=-\frac{1}{a^2}x+\frac{2}{a}\Leftrightarrow (x-a)^2=0\Leftrightarrow x=a$$

$\therefore$ the only intersect point is: $\left(a,\frac{1}{a}\right)$.

9, (i) Proof:

$$f(x)=(x+3)^5\ \Leftrightarrow\ f'(x)=5(x+3)^4(x+3)'=5(x+3)^4$$, on the other hand:

$$f(x)=(x+3)^5\ \Leftrightarrow\ f(x+3)=(x+6)^5\ \Leftrightarrow\ f'(x+3)=5(x+6)^4$$

(ii) Proof:

$$f(x+3)=x^5\ \Leftrightarrow\ f(x)=(x-3)^5\ \Leftrightarrow\ f'(x)=5(x-3)^4(x-3)'=5(x-3)^4$$

$$f(x+3)=x^5\ \Leftrightarrow\ f'(x+3)=5x^4$$

(iii) Proof:

$$f(x+3)=(x+5)^7\ \Leftrightarrow\ f(x)=(x+2)^7\ \Leftrightarrow\ f(x)=7(x+2)^6(x+2)'=7(x+2)^6$$

$$f(x+3)=(x+5)^7\ \Leftrightarrow\ f'(x+3)=7(x+5)^6(x+5)'=7(x+5)^6$$

15, Proof:

(a) $|f(x)|\le x^2\Rightarrow |f(0)|\le 0^2=0\Rightarrow f(0)=0$ also:

$$|f(x)|\le x^2\Rightarrow -x^2\le f(x)\le x^2\Leftrightarrow \frac{-x^2}{x}\le\frac{f(x)}{x}\le\frac{x^2}{x}\ (x\neq 0)$$

$$\because \lim_{x\to 0}\frac{-x^2}{x}=\lim_{x\to 0}(-x)=0=\lim_{x\to 0}x=\lim_{x\to 0}\frac{x^2}{x}$$

$$\therefore \lim_{x\to 0}\frac{f(x)}{x}=0$$

$$\therefore \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-0}{h}=\lim_{h\to 0}\frac{f(h)}{h}=0$$

$\therefore f$ is differentiable at 0.

(b) Obviously, if $|g(x)|\le x^2$, then $|f(x)|\le|g(x)|\le x^2 \Leftrightarrow$ same as problem (a) above.

Thus, one can say, as long as $|g(x)|\le x^2$ and $|f(x)|\le|g(x)|$, the function f is differentiable at 0.

23, Proof:

$let\ g(x)=f(-x),\ then\ g'(x)=\left(f(-x)\right)'=f'(-x)\cdot\left(-x\right)'=-f'(-x)$ (by chain rule)

on the other hand, f is even, thus, $f(x)=f(-x)\Leftrightarrow f(x)=g(x)\Leftrightarrow f'(x)=-f'(-x)$

Q.E.D.