Charpter 8

1, (ii) $\left\{\frac{1}{n}: n \text{ in } \mathbb{Z} \text{ and } n \neq 0.\right\}$

The least upper bound: 1, the greatest element: 1;

The greatest lower bound: 0, no least element.

(iv) $\left\{x : 0 \leq x \leq \sqrt{2} \text{ and } x \text{ is rational}.\right\}$

The least upper bound: $\sqrt{2}$, no greatest element;

The greatest lower bound: 0, the least element: 0.

(vi) $\left\{x : x^2 + x - 1 < 0\right\}$

The least upper bound: $\frac{-1+\sqrt{5}}{2}$, no greatest element;

The greatest lower bound: $\frac{-1-\sqrt{5}}{2}$, no least element.

(viii) $\left\{\frac{1}{n} + (-1)^n : n \text{ in } N.\right\}$

The least upper bound: $\frac{3}{2}$, the greatest element: $\frac{3}{2}$;

The greatest lower bound: -1, no least element.

6, Proof:

(a) We assume that: $\exists x_0$ s.t. $f(x_0) = a \neq 0$.

$\because$ f is contionous,

$\therefore \lim_{x \to x_0} f(x) = f(x_0) = a$

From the definition of limit, we know:

$$\exists d_0 > 0, s.t. \forall x : 0 < |x - x_0| < d_0, |f(x) - a| < \frac{|a|}{2}$$

$$|f(x) - a| < \frac{|a|}{2} \Rightarrow |f(x)| \geq \frac{|a|}{2} > 0, a \neq 0 \Rightarrow f(x) \neq 0 ,$$

also $f(x_0) \neq 0$

Then: $\exists d_0 > 0, s.t. \forall x : x_0 - d_0 < x < x_0 + d_0, f(x) \neq 0$ (*)

On the other hand, because A is a dense set, then $\exists\, x_a \text{ in } (x_0 - d_0, x_0 + d_0), s.t. x_a \in A$, It means: $f(x_a)=0$, which is conflict with (*), Q.E.D.

(b) Let $h(x)=f(x)-g(x)$, apply the conclusion of (a) to $h(x)$, we know $h(x)=0$ for all x, which means $f(x)=g(x)$ for all x. Q.E.D

(c) Likewise, let $h(x)=f(x)-g(x)$, then $h(x)$ is continuous and $h(x)\geq 0, \forall x \in A$. We assume that: $\exists x_0, s.t.\, h(x_0)=a<0$, then $\lim_{x\to x_0} h(x)=h(x_0)=a<0$.

$$\therefore\ \exists d_0>0, s.t.\ \forall x: 0<|x-x_0|<d_0, |h(x)-a|<\frac{|a|}{2}$$

$$|h(x)-a|<\frac{|a|}{2}\Rightarrow 0>\frac{a}{2}\geq h(x)\geq\frac{3a}{2}, a<0\Rightarrow h(x)<0$$

also $h(x_0)<0$

Then: $\exists d_0>0, s.t.\, \forall x: x_0-d_0<x<x_0+d_0, h(x)<0$ (*)

On the other hand, because A is a dense set, then $\exists\, x_a\ in\ (x_0-d_0, x_0+d_0), s.t.\ x_a\in A$, it means $h(x_a)\geq 0$, which is conflict with (*), Q.E.D.

$\geq$ can not be replaced by < throughout, cause even if $h(x)>0, \forall x\in A$, it is still a chance that $\lim h(x)=0$.

8, Proof:

(a) Considering with interval $(-\infty, a)$, since $f(a)\leq f(b), whenever\ a<b$, then for every $x\in(-\infty,a), f(x)\leq f(a)$, we can say $f(x)$ is bounded above at $(-\infty,a)$. From P13, we know there must exist a least upper bound of $f(x)$, let name it as K, obviously, $K\leq f(a)$. Now, we prove that $\lim_{x\to a-} f(x)=K$

K is a least upper bound, which means $\forall e>0, \exists x_0<a, s.t. K-f(x_0)<e$, then

Let $d=a-x_0$, obviously, $d>0$, and $\forall x\in(a-d,a), |f(x)-K|=K-f(x)$

$$\because\ f(a)\leq f(b), when\, a<b \Rightarrow f(x)\geq f(a-d), when\ x\in(a-d,a)$$

$$\therefore K-f(x)\leq K-f(a-d)=K-f(a-a-x_0)=K-f(x_0)<e$$

$\therefore \exists d = a - x_0 > 0, s.t. \forall x \in (a-d, a), |K - f(x)| < e$, which means:

$\lim_{x \to a-} f(x) = K$

Likewise, considering with interval (a, ∞), $f(x)$ has a greatest lower bound K', such that $\lim_{x \to a+} f(x) = K'$

(b) Assume that f has a removable discontinuity, then: $\lim_{x \to a} f(x) = K \neq f(a)$. Which means: $\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = K$, from above we know: $\lim_{x \to a-} f(x) \leq f(a) \leq \lim_{x \to a+} f(x)$

Which means $K \leq f(a) \leq K \Leftrightarrow f(a) = K = \lim_{x \to a} f(x) \Rightarrow\Leftarrow$ Q.E.D

(c) f satisfies the conclusions of the Intermediate Value Theorem, which means that: $\forall y \in (f(a), f(b)), \exists x \in [a, b] s.t. f(x) = y$ (*)

From (a), we know: $\forall c \in [a, b], both \lim_{x \to c-} f(x)\ and \lim_{x \to c+} f(x)\ exists$. Let $\lim_{x \to c-} f(x) = K_c\ and \lim_{x \to c+} f(x) = K'_c$.

Assume that $K_c \neq K'_c$, then:

from (a), we know: $f(a) \leq K_c \leq f(c) \leq K'_c \leq f(b)$, then:

$K_c \neq K'_c \Rightarrow K'_c > K_c$, let $d = K'_c - K_c$, obviously, $d > 0, also, \frac{d}{3} > 0$

$\therefore f(a) \leq K_c < K_c + \frac{d}{3} < K'_c - \frac{d}{3} < K'_c \leq f(b)$,

let $y_1 = K_c + \frac{d}{3}, y_2 = K'_c - \frac{d}{3}$, then:

$\because y_1 > K_c$ and K_c is a upper bound of set $\{f(x) : x \in [a, c)\}$

$\therefore y_1 \notin \{f(x) : x \in [a, c)\}$

$\because y_1 < K'_c$ and K'_c is a lower bound of set $\{f(x) : x \in (c, b]\}$

$\therefore y_1 \notin \{f(x) : x \in (c, b]\}$

$\therefore y_1 \notin \{f(x) : x \in [a, b], x \neq c\}$

Likewise, $y_2 \notin \{f(x): x \in [a,b], x \neq c\}$, and $y_1 \neq y_2$

Noticed that $y_1, y_2 \in [f(a), f(b)], \; and \; f(c) = y_1, f(c) = y_2$ can not be true at the same time, then:

$\exists y_0 (y_0 = y_1 \; or \; y_2), s.t. \; y_0 \in [f(a), f(b)], \; but \; y_0 \notin \{f(x): x \in [a,b]\}$,

which is conflict with (*) above.

Then, K_c has to be equal to K_c'. From (b) above, we know: if $K_c' = K_c$, then $\lim_{x \to c} f(x) = K = f(c)$, Q.E.D

13, Proof:

Since $x \le \sup(A), \; and \; y \le \sup(B) \Rightarrow x + y \le \sup(A) + \sup(B)$, then $\sup(A) + \sup(B)$ is an upper bound of $x + y$. Thus, $\sup(x+y) \le \sup(A) + \sup(B)$

On the other hand, $\because \forall e > 0, \exists x, s.t. \sup(A) - x < \frac{e}{2}, \; likewise, \exists y, s.t. \sup(B) - y < \frac{e}{2}$

$$\therefore \exists x + y, s.t. \forall e > 0, \sup(A) + \sup(B) - (x + y) < \frac{e}{2} + \frac{e}{2} = e$$

$$\Leftrightarrow \sup(A) + \sup(B) - e < (x + y) \le \sup(A + B)$$

Q.E.D.