Chapter 6

1. For which of the following functions f is there a continuous function F with domain $\Re$ such that $F(x) = f(x)$ for all x in the domain of f ?

I. $f(x) = \dfrac{x^2 - 4}{x - 2}$, Yes. There is such a function $g(x) = x + 2$

II. $f(x) = \dfrac{|x|}{x}$, No. Cause $\lim_{x \to 0^+} f \neq \lim_{x \to 0^-} f$

III. $f(x) = 0, \quad x\ irrational$. Yes, there is such a function $g(x) = 0$

IV. $f(x) = \dfrac{1}{q}, x = \dfrac{p}{q}$ rational in lowest terms. No. cause $\lim_{x \to a} f(x) \neq f(a)$

3. (a) suppose that f is a function satisfying $|f(x)| \le |x|$ for all x. Show that f is continuous at 0.

Solution:

$$\because |f(x)| \ge 0 \Rightarrow |f(0)| \ge 0$$
$$also \quad |f(x)| \le |x| \Rightarrow |f(0)| \le |0| = 0$$
$$\therefore |f(0)| = 0 \Leftrightarrow f(0) = 0$$
$$\because |f(x)| \le |x| \Leftrightarrow -|x| \le f(x) \le |x|$$

since we know $\lim_{x \to 0} |x| = 0$, $\lim_{x \to 0} (-|x|) = 0$

from a problem which we solved previously, if $g(x) \le f(x) \le h(x)$ and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x)$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x)$

$\therefore \lim_{x \to 0} f(x) = 0 = f(0)$, it's continuous at 0.

Q.E.D

(b) Give an example of such a function f which is not continuous at any $a \neq 0$.

Let $f(x) = \begin{cases} x = \dfrac{x}{2} & x\ irrational \\ x = -\dfrac{x}{2} & x\ rational \end{cases} \quad (x \neq 0)$

Obviously, it's not continuous at any $x \neq 0$

(c) Suppose that g is continuous at 0 and $g(0) = 0$, and $|f(x)| \le |g(x)|$. Prove that f is continuous at 0.

Solution:

Same as (a), $|f(x)| \le |g(x)| \Leftrightarrow -|g(x)| \le f(x) \le |g(x)|$

g is continuous at 0 and $g(0) = 0 \Leftrightarrow \lim_{x\to 0} g(x) = 0 \Leftrightarrow \lim_{x\to 0}|g(x)| = \lim_{x\to 0} -|g(x)| = 0$

$\therefore \lim_{x\to 0} f(x) = 0$

Also, $\left.\begin{array}{l} |f(x)| \le |g(x)| \Leftrightarrow |f(0)| \le |g(0)| = 0 \\ |f(x)| \ge 0 \end{array}\right\} \Rightarrow |f(x)| = 0 \Leftrightarrow f(x) = 0$

$\therefore \lim_{x\to 0} f(x) = 0 = f(0)$, it's continuous at 0.

12. (a) Prove that if f is continuous at l and $\lim_{x\to a} g(x) = l$, then $\lim_{x\to a} f(g(x)) = f(l)$.

Solution:

Construct a function G with $G(x) = g(x)\ for\ x \ne a, and\ G(a) = l$

$\lim_{x\to a} g(x) = l \Rightarrow \lim_{x\to a} G(x) = l$

$\therefore \lim_{x\to a} G(x) = G(a)$, G is continuous at a.

f is continuous at l and G is continuous at a. $\Rightarrow \lim_{x\to a} f(G(x)) = f(G(a)) = f(l)$

$\because G(x) = g(x)\ for\ x \ne a \Rightarrow \lim_{x\to a} f(G(x)) = \lim_{x\to a} f(g(x))$

$\therefore \lim_{x\to a} f(g(x)) = f(l)$

(b) Show that if continuity at l is not assumed, then it is not generally true that $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$.

For example:

Let $f(x) = 0\ for\ x \ne l, and\ f(l) = 1$, then obviously, for any $a \in \Re, \lim_{x\to a} f(x) = 0$

$\therefore \lim_{x\to a} f(g(x)) = 0$

On the other hand, $f(\lim_{x\to a} g(x)) = f(l) = 1$

$\therefore \lim_{x\to a} f(g(x))$ is not necessary to be equal to $f(\lim_{x\to a} g(x))$.

14. (a) Suppose that g and h are continuous at a, and that $g(a) = h(a)$. Define $f(x)$ to be $g(x)$ if $x \ge a$ and $h(x)$ if $x \le a$. Prove that $f(x)$ is continuous at a.

Prove:

First of all, obviously, $f(a) = g(a) = h(a)$

On the other hand, g and h are continuous at a $\Rightarrow \lim_{x \to a} h(x) = \lim_{x \to a} g(x) = g(a) = h(a)$

From the definition, we know

$\forall e > 0, \exists d_h > 0, s.t.\ for\ 0 < |x-a| < d_h, |h(x) - h(a)| < e$ (1)

also: $\forall e > 0, \exists d_g > 0, s.t.\ for\ 0 < |x-a| < d_g, |g(x) - g(a)| < e$ (2)

Let $d = \min(d_h, d_g)$

from (2), $for\ 0 < x - a < d$, $we\ know |f(x) - f(a)| = |g(x) - g(a)| < e$,

from (1), $for\ 0 > x - a > -d$, $we\ know |f(x) - f(a)| = |h(x) - h(a)| < e$,

that is: $\forall e > 0, \exists d > 0, s.t.\ for\ 0 < |x-a| < d, |f(x) - f(a)| < e$

$\therefore \lim_{x \to a} f(x) = f(a)$, it is continuous at point a.

(b) Suppose g is continuous on $[a,b]$ and h is continuous on $[b,c]$ and $g(b) = h(b)$. Let $f(x)$ be $g(x)$ for $x\ in\ [a,b]$ and $h(x)$ for $x\ in\ [b,c]$. Show that f is continuous on $[a,c]$.

Prove:

Obviously, $f(b) = g(b) = h(b)$

On the other hand, g and h are continuous at b $\Rightarrow \lim_{x \to b} h(x) = \lim_{x \to b} g(x) = g(b) = h(b)$

From the definition, we know

$\forall e > 0, \exists d_h > 0, s.t.\ for\ 0 < |x-b| < d_h, |h(x) - h(b)| < e$ (1)

also: $\forall e > 0, \exists d_g > 0, s.t.\ for\ 0 < |x-b| < d_g, |g(x) - g(b)| < e$ (2)

Let $d = \min(d_h, d_g)$

from (1), $for\ 0 < x - b < d$, $we\ know |f(x) - f(b)| = |h(x) - h(b)| < e$,

from (2), $for\ 0 > x - b > -d$, $we\ know |f(x) - f(b)| = |g(x) - g(b)| < e$,

that is: $\forall e > 0, \exists d > 0, s.t.\ for\ 0 < |x-b| < d, |f(x) - f(b)| < e$

$\therefore \lim_{x \to b} f(x) = f(b)$, it is continuous at point b.

it's obviously that $f(x)$ is continuous at anywhere else on [a, c]. Q.E.D.