Chapter 5

1, Find the following limits.

ii) $\lim_{x\to 2}\frac{x^3-8}{x-2}$

solution:

$$\lim_{x\to 2}\frac{x^3-8}{x-2}=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2}=\lim_{x\to 2}(x^2+2x+4)$$
$$=\lim_{x\to 2}x^2+\lim_{x\to 2}2x+\lim_{x\to 2}4=4+4+4=12$$

iv) $\lim_{x\to y}\frac{x^n-y^n}{x-y}$

solution:

$$\lim_{x\to y}\frac{x^n-y^n}{x-y}=\lim_{x\to y}\frac{(x-y)\left(\sum_{i=0}^{n-1}x^{n-1-i}y^i\right)}{x-y}=\lim_{x\to y}\left(\sum_{i=0}^{n-1}x^{n-1-i}y^i\right)$$
$$=\sum_{i=0}^{n-1}\lim_{x\to y}\left(x^{n-1-i}y^i\right)=\sum_{i=0}^{n-1}y^{n-1}=ny^{n-1}$$

vi) $\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}$

solution:

$$\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}=\lim_{h\to 0}\frac{\left(\sqrt{a+h}-\sqrt{a}\right)\left(\sqrt{a+h}+\sqrt{a}\right)}{h\left(\sqrt{a+h}+\sqrt{a}\right)}$$
$$=\lim_{h\to 0}\frac{h}{h\left(\sqrt{a+h}+\sqrt{a}\right)}=\lim_{h\to 0}\frac{1}{\sqrt{a+h}+\sqrt{a}}=\frac{1}{\lim_{h\to 0}\left(\sqrt{a+h}+\sqrt{a}\right)}$$
$$=\frac{1}{\lim_{h\to 0}\left(\sqrt{a+h}\right)+\lim_{h\to 0}\sqrt{a}}=\frac{1}{\sqrt{a}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$$

3, In each of the following cases, find a δ such that $|f(x)-l|<\varepsilon$ for all x satisfying $0<|x-a|<\delta$

ii) $f(x)=\frac{1}{x};\ a=1,\ l=1$

solution:

$$|f(x)-l|<\varepsilon\Leftrightarrow\left|\frac{1}{x}-1\right|<\varepsilon\Leftrightarrow\frac{|1-x|}{|x|}<\varepsilon\Leftrightarrow|1-x|<|x|\varepsilon\text{, since }\ |1-x|-1\le|x|,$$

then, $|1-x|<|x|\varepsilon\Leftarrow|1-x|<(|1-x|-1)\varepsilon\Leftrightarrow|1-x|+\varepsilon<|1-x|\varepsilon\Leftrightarrow|1-x|<\frac{\varepsilon}{1-\varepsilon}$

$\because |x-a| = |x-1|$

$\therefore let\ \delta = \dfrac{\varepsilon}{1-\varepsilon}, if\ 0 < |x-a| < \delta, then\ |f(x)-l| < \varepsilon\ for\ all\ x.$

iv) $f(x) = \dfrac{x}{1+\sin^2 x}; a = 0, l = 0$

solution:

$$|f(x)-l| < \varepsilon \Leftrightarrow \left|\frac{x}{1+\sin^2 x} - 0\right| < \varepsilon \Leftrightarrow |x| < |1+\sin^2 x| \cdot \varepsilon \text{, since } |1+\sin^2 x| \geq 1,$$

then $|x| < |1+\sin^2 x| \cdot \varepsilon \Leftarrow |x| < \varepsilon$

$\because |x-a| = |x-0| = |x|$

$\therefore let\ \delta = \varepsilon, if\ 0 < |x| < \delta, then\ |f(x)-l| < \varepsilon\ for\ all\ x.$

vi) $f(x) = \sqrt{x}; a = 1, l = 1$

solution:

$$|f(x)-l| < \varepsilon \Leftrightarrow |\sqrt{x}-1| < \varepsilon \Leftrightarrow |\sqrt{x}-1| \cdot |\sqrt{x}+1| < |\sqrt{x}+1| \cdot \varepsilon \Leftrightarrow |x-1| < |\sqrt{x}+1| \cdot \varepsilon$$

since $|\sqrt{x}+1| > 1$, then $|x-1| < |\sqrt{x}+1| \cdot \varepsilon \Leftarrow |x-1| < \varepsilon$

$\because |x-a| = |x-1|$

$\therefore let\ \delta = \varepsilon, if\ 0 < |x-a| < \delta, then\ |f(x)-l| < \varepsilon\ for\ all\ x.$

13. Suppose that $f(x) \leq g(x) \leq h(x)$ and that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. Prove that $\lim_{x\to a} g(x)$ exists, and that $\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = \lim_{x\to a} h(x)$.

Proof:

Let's consider two new function $T(x) = h(x) - f(x)\ and\ T'(x) = g(x) - f(x)$, since $f(x) \leq g(x) \leq h(x)$, then we know: $\forall x, T(x) > T'(x) > 0$.

On the other hand,

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) \Leftrightarrow \lim_{x\to a} f(x) - \lim_{x\to a} h(x) = 0 \Leftrightarrow \lim_{x\to a}\big(f(x) - h(x)\big) \Leftrightarrow \lim_{x\to a} T(x) = 0$$

from the definition of limit we know,

for $\forall \varepsilon > 0, \exists \delta_o, such\ that\ 0 < |x-a| < \delta_o \Rightarrow |T(x) - 0| < \varepsilon$ (1)

since $\forall x, T(x) > T'(x) > 0 \Rightarrow |T(x)| > |T'(x)|$, from (1) above, we know

for $\forall \varepsilon > 0, let\ \delta = \delta_o, then\ 0 < |x-a| < \delta \Rightarrow |T(x) - 0| < \varepsilon \Rightarrow |T'(x) - 0| < \varepsilon$, which means $\lim_{x \to a} T'(x) = 0$.

Then,

$$\lim_{x \to a} T'(x) = 0 \Leftrightarrow \lim_{x \to a}\left(g(x) - f(x)\right) = 0 \Leftrightarrow \lim_{x \to a}\left(g(x) - f(x)\right) + \lim_{x \to a} f(x) = \lim_{x \to a} f(x)$$

$$\Leftrightarrow \lim_{x \to a}\left[\left(g(x) - f(x)\right) + f(x)\right] = \lim_{x \to a} f(x) \Leftrightarrow \lim_{x \to a} g(x) = \lim_{x \to a} f(x)$$

Since we already knew $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$, then Q.E.D.

21.

a) Prove that if $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} g(x)\sin\frac{1}{x} = 0$.

Proof:

If $g(x) > 0$, $\left|\sin\frac{1}{x}\right| \le 1 \Leftrightarrow -1 \le \sin\frac{1}{x} \le 1 \Leftrightarrow -1 \cdot g(x) \le g(x) \cdot \sin\frac{1}{x} \le g(x)$

Since $\lim_{x \to a} g(x) = \lim_{x \to a} - g(x) = 0$, applying the result of problem 13 above, we know:

$$\lim_{x \to a} g(x)\sin\frac{1}{x} = 0$$

If $g(x) < 0$, $\left|\sin\frac{1}{x}\right| \le 1 \Leftrightarrow -1 \le \sin\frac{1}{x} \le 1 \Leftrightarrow -1 \cdot g(x) \ge g(x) \cdot \sin\frac{1}{x} \ge g(x)$,

same as $g(x) > 0$, we got $\lim_{x \to a} g(x)\sin\frac{1}{x} = 0$

If $g(x) = 0$, then $\lim_{x \to a} g(x)\sin\frac{1}{x} = \lim_{x \to a} 0 \cdot \sin\frac{1}{x} = \lim_{x \to a} 0 = 0$

Q.E.D.

b) Generalize this fact as follows: If $\lim_{x \to a} g(x) = 0$ and $|h(x)| \le M$ for all x, then

$$\lim_{x \to a} g(x)h(x) = 0$$

Proof:

If $g(x) > 0$,

$$|h(x)| \le M \Leftrightarrow -M \le h(x) \le M \Leftrightarrow -M \cdot g(x) \le g(x) \cdot h(x) \le M \cdot g(x)$$

Since $\lim_{x \to a} M \cdot g(x) = \lim_{x \to a} - M \cdot g(x) = M \cdot 0 = 0$, applying the result of problem 13 above, we know:

$$\lim_{x \to a} g(x)h(x) = 0$$

If $g(x) < 0$,

$$|h(x)| \le M \Leftrightarrow -M \le h(x) \le M \Leftrightarrow -M \cdot g(x) \ge g(x) \cdot h(x) \ge M \cdot g(x),$$

same as $g(x) > 0$, we got $\lim_{x \to a} g(x)h(x) = 0$

If $g(x) = 0$, then $\lim_{x \to a} g(x)h(x) = \lim_{x \to a} 0 \cdot h(x) = \lim_{x \to a} 0 = 0$

Q.E.D.

37. We define $\lim_{x \to a} f(x) = \infty$ to mean that for all N there is a $\delta > 0$, such that, for all x, if $0 < |x - a| < \delta, \textit{then } f(x) > N$.

a) Show that $\lim_{x \to 3} \frac{1}{(x-3)^2} = \infty$

Proof:

$$\frac{1}{(x-3)^2} > N \Leftrightarrow \frac{1}{N} > (x-3)^2 \Leftrightarrow |x-3| < \sqrt{\frac{1}{N}}$$

$$\therefore \textit{ for } \forall N, \exists\, \delta = \sqrt{\frac{1}{N}}, \textit{ such that if } 0 < |x-3| < \delta, \textit{ then } \frac{1}{(x-3)^2} > N$$

According to the definition, $\lim_{x \to 3} \frac{1}{(x-3)^2} = \infty$

b) Prove that if $f(x) > \varepsilon > 0$ for all x, and $\lim_{x \to a} g(x) = 0$ then

$$\lim_{x \to a} \frac{f(x)}{|g(x)|} = \infty$$

Proof:

$$\frac{f(x)}{|g(x)|} > N \Leftrightarrow f(x) > N \cdot |g(x)| \Leftrightarrow |g(x)| < \frac{f(x)}{N} \Leftarrow |g(x)| < \frac{\varepsilon}{N}$$

$\because \lim_{x \to a} g(x) = 0$

from the definition, $\exists \delta_0, \textit{ such that if } 0 < |x - a| < \delta_0, |g(x)| < \frac{\varepsilon}{N}$

$$\therefore \exists \delta_0, \textit{ such that if } 0 < |x - a| < \delta_0, \frac{f(x)}{|g(x)|} > N$$

Q.E.D.