Chapter 2

1, Prove the following formulas by induction.

(i) $1^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

Proof:

For n=1, we know $1^2=1$ *also* $\dfrac{1\cdot(1+1)(2\cdot 1+1)}{6}=\dfrac{6}{6}=1$ then, checked.

If for n = k, we know $1^2+\cdots+k^2=\dfrac{k(k+1)(2k+1)}{6}$, then

$$1^2+\cdots+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2=\frac{(k+1)(k(2k+1)+6(k+1))}{6}$$
$$=\frac{(k+1)(2k^2+7k+6)}{6}=\frac{(k+1)(k+2)(2k+3)}{6}=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

Q.E.D.

(ii) $1^3+\cdots+n^3=(1+\cdots+n)^2$

First of all, we prove that $1+\cdots+n=\dfrac{n(n+1)}{2}$

Proof: For n = 1, we know $1=\dfrac{1\cdot(1+1)}{2}=1$,

if n=k, we know $1+\cdots+k=\dfrac{k(k+1)}{2}$, then,

$$1+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)=\frac{k(k+1)+2(k+1)}{2}$$
$$=\frac{(k+1)(k+2)}{2}=\frac{(k+1)((k+1)+1)}{2}$$

thus, we already proved $1+\cdots+n=\dfrac{n(n+1)}{2}$

now let's go back to the $1^3+\cdots+n^3=(1+\cdots+n)^2$

For n = 1, we know $1^3=1$ *also* $1^2=1$, then, the equation is correct.

If for n = k, we know $1^3+\cdots+k^3=(1+\cdots+k)^2$, then

$$1^3+\cdots+k^3+(k+1)^3=(1+\cdots+k)^2+(k+1)^3$$
$$=(1+\cdots+k)^2+(k+1)(k+1)^2$$
$$=(1+\cdots+k)^2+k(k+1)^2+(k+1)^2$$
$$=(1+\cdots+k)^2+2\cdot\frac{(k+1)k}{2}(k+1)+(k+1)^2$$
$$=(1+\cdots+k)^2+2\cdot(1+\cdots+k)(k+1)+(k+1)^2=(1+\cdots+k+(k+1))^2$$

Q.E.D.

5. Prove by induction on n that

$$1+r+r^2+\cdots+r^n=\frac{1-r^{n+1}}{1-r} \quad \text{if } r\neq 0$$

Proof:

For n = 1, $\frac{1-r^2}{1-r}=\frac{(1-r)(1+r)}{(1-r)}=(1+r) \quad r\neq 0$, equation checked.

Assume for n = k, $1+r+r^2+\cdots+r^k=\frac{1-r^{k+1}}{1-r}$, then

$$1+r+r^2+\cdots+r^k+r^{k+1}=\frac{1-r^{k+1}}{1-r}+r^{k+1}=\frac{1-r^{k+1}+(1-r)r^{k+1}}{1-r}$$

$$=\frac{1-r^{k+1}+r^{k+1}-r^{k+2}}{1-r}=\frac{1-r^{(k+1)+1}}{1-r} \quad r\neq 1$$

Q.E.D.

(b) Derive this result by setting S $=1+r+r^2+\cdots+r^n$, multiplying this equation by r, and solving the two equation for S.

Solve:

$$let \quad S=1+r+r^2+\cdots+r^n, \quad then\ S\cdot r=(1+r+\cdots+r^n)r=r+r^2+\cdots+r^n+r^{n+1}$$
$$=S-1+r^{n+1}$$

$$\therefore S\cdot r=S+r^{n+1}-1\Leftrightarrow S\cdot(r-1)=r^{n+1}-1\Leftrightarrow S=\frac{r^{n+1}-1}{r-1}=\frac{1-r^{n+1}}{1-r} \quad r\neq 1$$

Chapter 3

6. (a) If $x_1,x_2,\cdots,x_n$ are distinct numbers, find a polynomial function f_i of degree n-1 which is 1 at x_i and 0 at x_j for $j\neq i$.

Solve:

$$Let \quad \partial(x)=\prod_{\substack{j=1\\ j\neq i}}^{n}(x-x_j), \text{ we have } \partial(x_k)=0, \quad for\ \forall k\neq i$$

$$\therefore let \quad \sigma(x)=\frac{\partial(x)}{\partial(x_i)} \quad then \quad \sigma(x_i)=1\ and\ \sigma(x_k)=0\ for\ k\neq i$$

Q.E.D.

(b) Now find a polynomial function f of degree n-1 such that $f(x_i)=a_i$ $where\ a_1,\cdots,a_n$ are given numbers.

Solve:

$$let \quad \delta(x) = \sum_{j=1}^{n} a_j \frac{\prod_{\substack{i=1 \\ i \neq j}}^{n} (x - x_i)}{\prod_{\substack{i=1 \\ i \neq j}}^{n} (x_j - x_i)}$$, then $\delta(x_i) = a_i$

13. (a) Prove that function f with domain $\Re$ can be written $f = E + O$ where E is even and O is odd.

Solve:

$$Let \quad E(x) = \frac{f(x) + f(-x)}{2} \quad O(x) = \frac{f(x) - f(-x)}{2} \quad then$$
$$f(x) = E(x) + O(x)$$
$$E(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = E(x) \quad E \text{ is even.}$$
$$O(-x) = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -O(x) \quad O \text{ is odd}$$
$$Q.E.D.$$

(b) Prove that this way of writing f is unique.

Proof:

$$\text{assume there are more than one way to do so, then there must be}$$
$$f = E + O \text{ and } f = E' + O' \quad \text{in which } E \text{ and } E' \text{ are even, } O \text{ and } O' \text{ are odd}$$
$$\text{then } E(-x) = E(x), E'(-x) = E'(x), O(-x) = -O(x), O'(-x) = -O'(x)$$
$$\therefore f(x) + f(-x) = E(x) + O(x) + E(-x) + O(-x)$$
$$= E(x) + O(x) + E(x) - O(x) = 2E(x)$$
$$\text{also } f(x) + f(-x) = E'(x) + O'(x) + E'(-x) + O'(-x) = 2E'(x)$$
$$\text{thus } E'(x) = E(x)$$
$$\text{similarly, we can prove that } O'(x) = O(x).$$
$$Q.E.D.$$