UNIVERSITY OF TORONTO

Faculty of Arts and Sciences

APRIL/MAY EXAMINATIONS 2003

Math 157Y Analysis I — Final Exam

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Name: ______________________________ Student ID: __________________

Solve the following 6 problems. Each is worth 20 points although they may have unequal difficulty, so the maximal possible total grade is 120 points. Write your answers in the space below the problems and on the front sides of the extra pages; use the back of the pages for scratch paper. Only work appearing on the front side of pages will be graded. Write your name and student number on each page. If you need more paper please ask the presiding officers. This booklet has 12 pages.

Duration. You have 3 hours to write this exam.

Allowed Material: Any calculating device that is not capable of displaying text.

Good Luck!

For Grading Use Only

1	/20	4	/20
2	/20	5	/20
3	/20	6	/20
Total	/120		

web version: http://www.math.toronto.edu/~drorbn/classes/0203/157AnalysisI/Final/Exam.html

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Problem 1. Let f and g denote functions defined on some set A.

1. Prove that
$$\sup_{x\in A}(f(x)+g(x)) \le \sup_{x\in A} f(x) + \sup_{x\in A} g(x).$$

2. Find an example for a pair f, g for which
$$\sup_{x\in A}(f(x)+g(x)) = \sup_{x\in A} f(x) + \sup_{x\in A} g(x).$$

3. Find an example for a pair f, g for which
$$\sup_{x\in A}(f(x)+g(x)) < \sup_{x\in A} f(x) + \sup_{x\in A} g(x).$$

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Problem 2. Sketch the graph of the function $y = f(x) = \dfrac{x^2}{x^2 - 1}$. Make sure that your graph clearly indicates the following:

- The domain of definition of $f(x)$.
- The behaviour of $f(x)$ near the points where it is not defined (if any) and as $x \to \pm\infty$.
- The exact coordinates of the x- and y-intercepts and all minimas and maximas of $f(x)$.

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Problem 3. Compute the following integrals:

1. $\displaystyle\int \frac{x^2+1}{x+1}dx.$

2. $\displaystyle\int \frac{x+1}{x^2+1}dx.$

3. $\displaystyle\int x^2 \sin x \, dx.$

4. $\displaystyle\int \frac{dx}{\sqrt{1+e^x}}.$

5. $\displaystyle\int_0^\infty e^{-x} dx.$

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Problem 4. Agents of the CSIS have secretly developed a function $e(x)$ that has the following properties:

- $e(x+y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.
- $e(0) = 1$
- e is differentiable at 0 and $e'(0) = 1$.

Prove the following:

1. e is everywhere differentiable and $e' = e$.
2. $e(x) = e^x$ for all $x \in \mathbb{R}$. The only lemma you may assume is that if a function f satisfies $f'(x) = 0$ for all x then f is a constant function.

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Problem 5.

1. Prove that if a sequence of continuous functions f_n converges uniformly to a function f on some interval $[a, b]$, then f is continuous on $[a, b]$.
2. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{2^n} \sin(3^n x)$ converges on $(-\infty, \infty)$ and that its sum is a continuous function of x.

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Problem 6. Prove that the complex function $z \mapsto \bar{z}$ is everywhere continuous but nowhere differentiable.