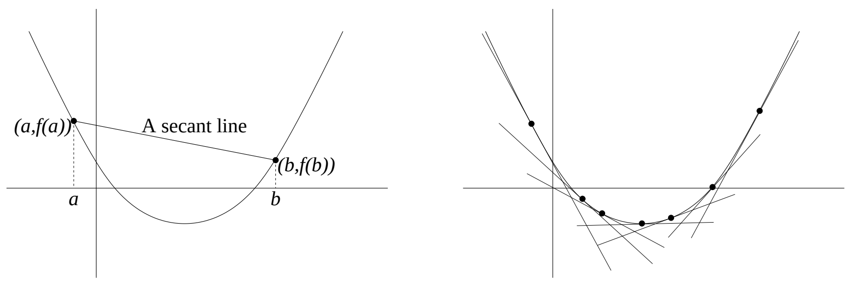
Dror Bar-Natan: Classes: 2002-03: Math 157 - Analysis I:

A Little on Convexity

web version:
www.math.toronto.edu/~drorbn/classes/0203/157AnalysisI/Convexity/Convexity.html

We are skipping the appendix on convexity of Spivak's Chapter 11, but it is still worthwhile to take something from it (without proof):



Theorem. The following are equivalent, for a function f defined on some interval I (assuming f is such that these statements make sense):

1. All the secants of f are above the graph of f.
2. For every $a, b \in I$ and every $t \in (0,1)$,
$$f(ta + (1-t)b) < tf(a) + (1-t)f(b).$$
3. The tangents to the graph of f all lie below that graph and touch it just at the points of tangency.
4. The derivative f' is increasing.
5. The second derivative f'' is positive on I: $\forall x \in I \; f''(x) > 0$. (Gary Baumgartner makes the following correction: This last statement implies all others, but it isn't implied by the others as can be seen by looking for example at $f(x) = x^4$. If all sharp inequalities in this handout are replaced by non-sharp ones (i.e., replace $>$ by $\geq$ and $<$ by $\leq$ everywhere, with similar corrections for verbal statements), then this statement becomes equivalent to all others).

If any of these statements holds, we say that "f is convex". There is a similar theorem with all inequalities reversed, and then the name is "f is concave".