FUNDAMENTAL CONCEPTS IN DIFFERENTIAL GEOMETRY
FALL 2000
HANDOUT # 2

1. Exercises for the Proper Course

Exercises 1 and 2 are extracted from Bredon's book.

1. Let X be the graph of the real valued function $\theta(x) = |x|$ of a real variable x. Define a functional structure on X by taking $f \in F(U)$ if and only if f is the restriction to U os a C^∞ function on some open subset V of $\mathbb{R}^2$ such that $U = V \cap X$. Show that X with this structure is *not* diffeomorphic to the real line with the usual C^∞ structure.

2. Let X be a copy of the real line $\mathbb{R}$ and let $\phi(x) = x^3$. Taking ϕ as a chart, this defines a smooth structure on X. Prove or disprove the following statements:

(1) X is diffeomorphis with $\mathbb{R}$.
(2) the identity map $X \to \mathbb{R}$ is a diffeomoephism.
(3) ϕ together with the identity map comprise an atlas.
(4) on the one point compactification X^+ of X, ϕ and ψ give an atlas, where $\psi(x) = 1/x$, for $x \neq 0$, and $\psi(\infty) = 0$. (ψ is defined on $X^+ - \{0\}$.)

3. The space $\mathbb{C}P^n$ is the quotient of $\mathbb{C}^{n+1} - \{0\}$ under the equivalence relation

$$(z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n) \qquad , \lambda \in \mathbb{C}^*.$$

Let $\pi : \mathbb{C}^{n+1} \to \mathbb{C}P^n$ be the projection map. Define the smooth structure using the pushforward of the structure sheaf. Show that so defined $\mathbb{C}P^n$ is a smooth manifold.

Find a diffeomorphism of S^2 and $\mathbb{C}P^1$.

4. Show that every connected 1-dimensional smooth manifold is diffeomerphic to the unit circle S^1. You may assume that your manifold is given a metric d if you find it convenient.

Date: 7 Nov., 2000.