Lecture 6

Combinatorial Methods (Winter 2023)
University of Toronto
Swastik Kopparty
Scribe: Lora Hreish, Volodymyr Khrystynych

1 Ramsey Theory for regular graphs

Theorem 1. *Take the complete bipartite graph $K(n,n)$ with n vertices, $n \geq 1000$, on either side. Color all edges red or blue arbitrarily. Then, there is a monochromatic $K_{2,2}$.*

Proof. Number of red edges or number of blue is $1/2n^2$. So by theorem from last time and since $1/2n^2 \geq O(n^3/2)$, there is $K_{2,2}$ in the more popular color. □

Theorem 2. *For $n \geq 1000$. Take K_n (the complete graph on n vertices). Color all the edges red or blue arbitrarily. Then, there is a monochromatic K_3 (triangle).*
Q: Does $\geq 1/2\binom{n}{2}$ edges in an n-vertex graph guarantee a triangle?
A: No. $K_{2,2}$ has $n^2/4$ edges but has no triangle. So previous reasoning doesn't work as most popular color can be in $K_{2,2}$.

Proof. For n = 6.
Take a vertex v. v has 5 edges (in K_6). ≥ 3 edges are of the same color, say red. Let u_1, u_2, u_3 be 3 neighbors of v such that vu_i is red. Then, if any u_iu_j is red then vu_iv_j is a red triangle. Else, $u_1u_2u_3$ is a blue triangle. □

Theorem 3. *Ramsey's theorem: For any number of colors c and for any size of clique k, there exists n_0 such that for every $n \geq n_0$ and for any c-coloring of the edges of K_n, there exists some monochromatic K_k.*

Proof. Assume $n \geq TBD$.
Pick any vertex v. There are $n-1$ edges from v. So $\geq \frac{n-1}{c}$ edges of the same color, α.
Let $S = \{$ vertices joined to v by color α $\}$.
$|S| \geq \frac{n-1}{c}$.
Claim: $P(k_1, ..., k_c)$ = "$\forall k_1, ..., k_c, \exists n_0(k_1, ..., k_c)$ s.t. $\forall n \geq n_0$, any c-coloring of edges of K_n has either

- K_{k_1} in color 1.
- K_{k_2} in color 2.
  ⋮
- K_{k_c} in color c. "

If $\left|S\right| \geq n_0(k_1, ...k_{\alpha-1}, k_\alpha - 1, k_{\alpha+1}, ..., k_c)$ then there exists either:

- K_{k_1} in color 1 (done)
- K_{k_2} in color 2 (done)
  $\vdots$
- $K_{k_{\alpha-1}}$ in color $\alpha - 1$ (done).
- $K_{k_\alpha - 1}$ in color α. (done with v included $\Rightarrow$ gives K_{k_α}).
  $\vdots$
- K_{k_c} in color c (done).

Value of $n_0(k_1, ..., k_c) = 1 + c.max_\alpha n_0(k_1, ..., k_{\alpha-1}, k_\alpha - 1, k_{\alpha+1}, ..., k_c)$.
Can take $n_0(k_1, ..., k_c) = 1 + \sum_\alpha n_0(k_1, ..., k_{\alpha-1}, k_\alpha - 1, k_{\alpha+1}, ..., k_c)$ □

Theorem 4. *Infinite (countable) version of Ramsey's Theorem: For a complete graph on countable infinite number of vertices and for $c \in \mathbb{N}$ colors, there exists an infinite countable monochromatic complete subset of the graph.*

Proof. Start with a vertex v_0. $\exists$ some color α_0 s.t. there are infinitely many edges of v_0 colored α_0. Let S_0 be the set of neighbors of v_0 with edges of color α_0.
Now move into S_0. Pick $v_1 \in S_0$. There are infinitely many edges from v_1 to S_0 and at least infinitely many are of the same color, say α_2. Let $S_1 = \{u \in S_0 \big| \; v_1 u \text{ is colored } \alpha_1\}$, $\left|S_1\right| = \infty$.
Repeat.
We get sequences:

- $v_0, v_1, ...,$ with $v_i \in S_{i-1}$.
- $\alpha_0, \alpha_1, ...,$ s.t. $\alpha_i \in [c]$.
- $S_0, S_1, ...,$ with $S_j \subset S_i$ for $j > i$.

Also note that color $(v_i v_j) = \begin{cases} \alpha_i, \text{ if } i < j \\ \alpha_j, \text{ if } i < i \end{cases}$
Since there are finitely many α_i, some color β appears as α_i for infinitely many i.
Let $I = \{i \text{ s.t. } \alpha_i = \beta\}$.
$V = \{v_i \text{ s.t. } i \in I\}$.
color(v_i, v_j) for $i, j \in I$ is equal to α_i or α_j which is β. □

2 Ramsey for a 3-uniform hypergraph

Definition 5. ***Hypergraph***: *(V, E) where the set E is composed of hyperedges, $E \subseteq P(V)$*

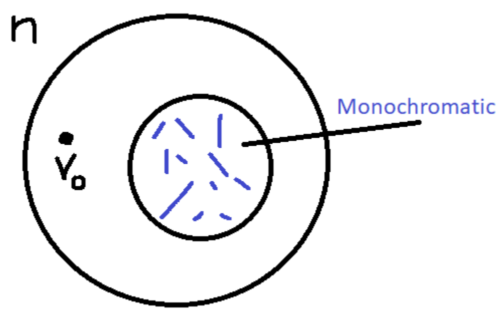
Definition 6. ***3-Uniform Hypergraph***: *(V, E) where the hyperedges are $E \subseteq \binom{V}{3}$*

Definition 7. *$K_n^{(3)}$: the complete 3-uniform hypergraph on n vertices*

Theorem 8. *For all c,k there exists an n_0 such that for all $n \geq n_o$ any colouring of the edges of $K_n^{(3)}$ has a monochromatic $K_k^{(3)}$.*

2.1 Proof 1: Using Graph Ramsey into Pigeonhole Principle

Proof. Create graph G_0 on $[n] \setminus v_0$ colour ab in $K_n^{(3)}$. By Ramsey's theorem for graphs, if $n-1$ is big enough then there exists some S_0 such that all edges in S_0 are coloured some colour α_0.



So every hyperedge v_0ab with $a, b \in S_0$, where $|S_0| \approx \mathcal{O}(\log n)$ and the hyperedge v_0ab has colour α_0. Take any $v_1 \in S_0$ create a colouring of $K_{S_0 \setminus v_0}$ by colouring in $K_k^{(3)}$ of vab. There exists a monochromatic clique S_1 by graph Ramsey with colour α_1. So any hyperedge v_1ab with $a, b \in S_1$ has colour α_1. With $|S_1| \approx \log |S_0|$. Repeating many times, we get:

$$\begin{array}{cccc} v_0 & v_1 & v_2 & ...vertices \\ \alpha_0 & \alpha_1 & \alpha_2 & ...colours \\ S_0 & S_1 & S_2 & ...|S_i| \approx \mathcal{O}(\log |S_{i-1}|) \end{array}$$

1. $S_j \subseteq S_i, \forall i < j$
2. $v_i \in S_{i-1}$
3. $\text{colour}(v_iab) = \alpha_i, \forall a, b \in S_i$

These 3 facts together means that $\text{colour}(v_iv_jv_k) = \alpha_i$, $\forall i < j < k$. Repeat this process ck times, some colours appear as α_i for at least k choices of i. $I = \{i \text{ such that } \alpha_i \in \beta \}$. Then if $|I| \geq k$, $\text{colour}(v_iv_jv_k) = \beta$, $\forall i, j, k \in I$ This means that $v_i : i \in I$ is the desired $K_k^{(3)}$. Now because

$|S_i| \approx \mathcal{O}(\log |S_{i-1}|$ we took $\log_2 ck$ times in order to get the desired $K_k^{(3)}$ and we need to reverse that in order to get the upper limit. This means

$$R^{(3)}(K) \leq 2^{2^{2^{\cdot^{\cdot}}}}$$

Where $R^{(3)}(K)$ is the minimum size of a graph that a random c colouring will have a monochromatic $K_k^{(3)}$ and $2^{2^{2^{\cdot^{\cdot}}}}$ is a power tower of height ck. □

Here we used graph Ramsey at every stage to refine the set then used the pigeonhole principle at the end. It's possible to use the pigeonhole principle to refine and graph Ramsey at the end.

2.2 Proof 2: Using Pigeonhole Principle into Graph Ramsey

Proof. Start with a pair v_0v_1 consider v_0v_1a. Some colour α_0 is most popular. Zoom into that:

$$S_0 = \{a : colour(v_0v_1a) = \beta\}$$

$$|S_0| \geq \frac{n-2}{c}$$

Pick $v_2 \in S_1$. For each $b \in S_0$ consider the tuple of colours $(v_0v_2b, v_1v_2b) \in [C]^2$. Let $(\alpha_{02}, \alpha_{12}) \in [C]^2$ be the popular colours seen and let $S_2 = \{b : colour(v_0v_2b) = \alpha_{02}, colour(v_1v_2b) = \alpha_{12}\}$

$$|S_2| \geq \frac{|S_1|}{c^2}$$

Why c^2? Because there are c^2 possible colours.

$$\begin{array}{ccccc} v_0 & v_1 & v_2 & ...v_{l-1} & \\ \alpha_{01} & \alpha_{02} & \alpha_{12} & ...\alpha_{l-1} & \forall i < j < l \\ S_1 & S_2 & S_3 & ...S_{l-1} & \end{array}$$

$$colour(v_iv_jb) = \alpha_{ij}, \forall b \in S_j, i < j$$

Pick $v_l \in S_{l-1}$. For each $b \in S_{l-1}$ consider colour vector

$$\vec{w}(b) = (colour(v_0v_lb), colour(v_1v_lb), colour(v_2v_lb), ..., colour(v_{l-1}v_lb))$$

Take the most popular colour vector

$$(\alpha_{0l}, \alpha_{1l}, \alpha_{2l}, ..., \alpha_{l-1,l})$$

Let

$$S_l = \{b : \vec{w}(b) = (\alpha_{0l}, \alpha_{1l}, \alpha_{2l}, ..., \alpha_{l-1,l})\}$$

$$|S_l| \geq \frac{|S_{l-1}|}{c^l}$$

Repeat: we have

$$v_0, v_1, ..., v_i, ..., v_m, \alpha_{ij} \in [C], m \geq R(k)$$

Apply Graph Ramsey

$$colour(v_i v_j v_k) = \alpha_i j \in [C], i < j < k$$

If V is the mono clique of size k then $colour(v_i v_j v_k)$ is the same colour which means that we have a monochromatic $K_k^{(3)}$. $|S_i| = \frac{|S_{i-1}|}{c^i}$, we need i to go into $R_c(k)$

$$|S_{R_c(k)}| = \frac{n}{c^{0+1+2+3+...+R(k)}} = \frac{n}{c^{R_c(k)}} > 1$$

$$n \geq c^{R_c(k)} = c^{c^{c^k}}$$

□

2.3 Minimum size of $R_2(k)$

Definition 9. $R_c(k)$ *= smallest* n *such that any* c *colouring of* K_n *has a monochromatic* K_k

Definition 10. $R_c^{(3)}(k)$ *= smallest* n *such that any* c *colouring of* $K_n^{(3)}$ *has a monochromatic* $K_k^{(3)}$

We saw: $R_c(k) \leq c^{O(ck)}$ and $R_c^{(3)}(k) \leq c^{c^{O(ck)}}$. In the 1930s there was a conjecture by Erdős and Szekeres that $R_2(k) \leq k^2$. In the 1940s Erdős showed that $R_2(k) \geq (\sqrt{2})^k$

Theorem 11. $R_2(k) \geq (\sqrt{2})^k$

Proof. Proof that $R_2(k) \geq (\sqrt{2})^k$: Set $n \geq TBD$, take a random 2-colouring of K_n. We will show with probability greater than 0 that the resulting colouring has no monochromatic k-clique.

$$\begin{aligned}\mathbb{P}[\text{There exists a monochromatic red k-clique}] &\leq \sum_{S \subseteq \binom{[n]}{k}} \mathbb{P}[\text{S is a red clique}] \\ &\leq \binom{n}{k} * \frac{1}{2^{\binom{k}{2}}}\end{aligned}$$

Choose n so that $2^{\binom{k}{2}} < 1/10$

$$\binom{n}{k} < \frac{2^{\binom{k}{2}}}{10}$$

Using a useful inequality: $\binom{n}{k} < (\frac{en}{k})^k$

$$\begin{aligned}(\frac{en}{k})^k &< \frac{2^{\binom{k}{2}}}{10} \\ \frac{en}{k} &< \frac{2^{\frac{k-1}{2}}}{10} \\ n &< \frac{k * 2^{\frac{k-1}{2}}}{10e} < O(k * e^{k/2})\end{aligned}$$

$$\begin{aligned}\mathbb{P}[\text{There exists a monochromatic clique}] &\leq \mathbb{P}[\text{There exists red k-clique}] + \mathbb{P}[\text{There exists blue k-clique}] \\ &\leq 1/10 + 1/10 < 1\end{aligned}$$

$\square$