Lecture 8: Absolute Spectral Expansion and Tanner Expander Codes

Topics in Error-Correcting Codes (Fall 2022)
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1 Spectral Expansion

Consider a d-regular graph $G = (V, E)$ with n vertices. Let A be the adjacency matrix of G. Consider the eigenvalues of A. Note that A is a symmetric matrix therefore all of the eigenvalues are real. Recall that an eigenvalue λ is a constant such that $A\vec{x} = \lambda\vec{x}$. There are at most n eigenvalues. We will call them $\lambda_1, \lambda_2, \ldots, \lambda_n$. Since they are all real, we can order them.

Claim 1. $\lambda_1 = d$

Proof. First, show d is an eigenvalue. Note that for the vector of all 1's, d is an eigenvalue because the vth entry of $A\vec{x}$ is $\sum_{u\sim v} x_u$.

Now, show that d is the max eigenvalue. If $A\vec{x} = \lambda\vec{x}$, the uth entry for $A\vec{x}$ is the sum of all values around it.

$$A\vec{x} = \lambda\vec{x} \Rightarrow \lambda x_u = \sum_{u\sim v} x_u$$

If $x_u = \max_{w\in V}(x_w) \Rightarrow \lambda x_u = \sum_{\max} x_u \leq d x_u$. □

Since A is symmetric, $\exists \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in \mathbb{R}^n$ nonzero such that

1. All $\vec{v}_1, \ldots, \vec{v}_n$ are orthogonal
2. $A\vec{v}_i = \lambda_i \vec{v}_i$

Lemma 2. $\lambda_2 = d$ *if and only if* G *is disconnected.*

Proof. First, assume G is disconnected. Then $v_1 = 1_{\text{component 1}}$ and $v_2 = 1_{\text{component 2}}$. Both are eigenvectors of d.

Now, assume $\lambda_2 = d$. Consider $\vec{v}_1 = \vec{1}$ and $\vec{v}_2 =\perp \vec{1}$ both eigenvectors of d.

If $x : V \to \mathbb{R}$ is an eigenvector of d, then the vth component of Ax equals $\sum_{u\sim v} x_u = d x_u$. Therefore we have

$$x_u = \frac{1}{d}\sum_{u\sim v} x_u \Rightarrow \text{avg } x_n \text{ neighbours of } V$$
$$\Rightarrow x \text{ is constant on connected components.}$$

So $\vec{v}_2$ is not all constant, therefore G is disconnected.

$\square$

Definition 3. *A d-regular graph is a λ-spectral expander if $\lambda_2 \leq \lambda$.*

An interesting consideration is $\lambda \leq (0.9)d$. Note that λ can go as small as $O(\sqrt{d})$.

Lemma 4. *1. $\lambda_n \geq -d$*

2. $\lambda_n = -d$ if and only if G is bipartite.

Proof. 1. $\lambda_n x_v = \sum_{u \sim v} x_u$. $x_v = \frac{1}{\lambda_n} \sum_{u \sim v} x_u$.

2. The proof of number 2 can be found online at various sources [1]. In general, the proof goes by comparing absolute value of the eigenvector in a certain coordinate with the absolute values of the eigenvector at all the coordinates of the neighbors.

$\square$

Definition 5. *A d-regular graph is a λ-absolute spectral expander if λ_2, $|\lambda_n| \leq \lambda$.*

Again, an interesting consideration is $\lambda \leq (0.9)d$. Note that λ can go as small as $O(\sqrt{d})$.

Lemma 6. *Assume G is a λ-absolute spectral expander. Let $S, T \subseteq V$ and*

$$\begin{aligned} e(S,\, T) &= \textit{number of edges between } S,\, T \\ &= \textit{ number of } (s,\, t) \textit{ such that } s \in S,\, t \in T, \ \textit{ and } s, t \textit{ is an edge.} \end{aligned}$$

Then, we have that

$$\left| e(S,\, T) - \frac{|S||T|d}{n} \right| \leq \lambda \sqrt{|S||T|}$$

Proof. Consider 1_S, 1_T. Then $e(S,\, T) = 1_T^\intercal A 1_S$ We can write

$$1_S = \sum_{i=1}^{n} \alpha_i v_i$$

$$1_T = \sum_{i=1}^{n} \beta_i v_i$$

[1] https://math.stackexchange.com/questions/3636376/eigenvalues-of-k-regular-bipartite-graph-adjacency-matrix/3636551#3636551

When $A1_S = \sum_i \alpha_i A v_i = \sum_i \alpha_i \lambda_i v_i$. We have

$$
\begin{aligned}
\langle 1_T,\, A1_S \rangle &= \langle \sum_j \beta_j v_j,\, \sum_i \alpha_i \lambda_i v_i \rangle \\
&= \sum_i \alpha_i \beta_i \lambda_i \\
&= \alpha_1 + \beta_1 d + \sum_{i=2}^{n} \alpha_i \beta_i \lambda_i \\
&= \frac{|S|}{\sqrt{n}} \frac{|T|}{\sqrt{n}} + \sum_{i=2}^{n} \alpha_i \beta_i \lambda_i
\end{aligned}
$$

Note that since

$$
\sum_i \alpha_i^2 = ||1_S||_2^2
$$

$$
a_i = \langle 1_s, v_i \rangle \Rightarrow a_i = \frac{|S|}{\sqrt{n}}
$$

we have that

$$
\begin{aligned}
\left| \sum_{i=2}^{n} \alpha_i \beta_i \lambda_i \right| &\leq \lambda \sum_{i=2}^{n} |\alpha_i \beta_i| \\
&\leq \lambda \left(\sum_i \alpha_i^2 \right)^{\frac{1}{2}} \left(\sum_i \beta_i^2 \right)^{\frac{1}{2}} \\
&\leq \lambda \left(|S| - \frac{|S|^2}{n} \right) \left(|T| - \frac{|T|^2}{n} \right) \\
at70 &\leq \lambda \sqrt{|S| - |T|} \sqrt{\left(1 - \frac{|S|}{n}\right)\left(1 - \frac{|T|}{n}\right)}
\end{aligned}
$$

□

2 Expander Graphs (Second Half)

Definition 7. *Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of a d-regular graph G with n vertices and d edges. We call G a λ-*spectral expander *if $\lambda \geq \lambda_2$.*

Comment. λ can go as small as $\mathcal{O}(\sqrt{d})$.

Theorem 8. *In a d-regular graph with n vertices, the least eigenvalue λ_n of the symmetric matrix satisfies $\lambda_n \geq -d$. The equality $\lambda_n = -d$ holds if and only if G is bipartite.*

Consider $\lambda_n \chi_v = \sum_{u \sim v} \chi_u$ where $\chi : V \longrightarrow \mathbb{C}$ is the map from the vertex set to the complex numbers. The idea of the proof is to take 1 on one component and -1 on the other.

Definition 9. *An absolute λ-spectral expander is a d-regular graph such that both $\lambda_2, |\lambda_n| \leq \lambda$.*

Comment. An interesting regime is $\lambda = 0.9d$.

Lemma 10 (Expander Mixing Lemma)**.** *Given subsets S and T of the vertex set V not necessarily disjoint, denote the number of edges between S and T by*

$$e(S,T) = \#\{(s,t) : s \in S, t \in T, st \text{ an edge}\}$$

In a random graph we have

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \leq \lambda \sqrt{|S||T|}$$

Proof. Let $\nVdash_S$ be the vector indicator of S and let A be the adjacency matrix of the graph G. Then we have $e(S,T) = \nVdash_T^\top A \nVdash_S^\top$. Put $\nVdash_S = \sum \alpha_i v_i$ and $\nVdash_T = \sum \beta_i v_i$ for orthogonal unit eigenvectors $\{v_i\}$ of the adjacency matrix. On the one hand we have $A\nVdash_S = \sum \alpha_i \lambda_i v_i$ and on the other we can compute

$$\left[e(S,T) = \langle \nVdash_T, A\nVdash_S \rangle = \langle \sum \beta_i v_i, \sum \alpha_i \lambda_i v_i \rangle = \sum \alpha_i \beta_i \lambda_i = \tfrac{d}{n}|S||T| + \underbrace{\sum_{i=2}^{n} \alpha_i \beta_i \lambda_i}_{\varepsilon} \right]$$

Now $\sum \alpha_i^2 = ||\nVdash_S||^2$ where $\alpha_i = \langle \nVdash_S, v_i \rangle$ so $\alpha_i = |S|/\sqrt{n}$. Analogous equation holds for β_i. So for the error term we have, by Cauchy-Schwartz

$$\varepsilon = \sum_{i=2}^{n} \alpha_i \beta_i \lambda_i \leq \lambda \sum |\alpha_i \beta_i| \leq \lambda \left(|S| - \frac{|S|^2}{n} \right)^{1/2} \left(|T| - \frac{|T|^2}{n} \right)^{1/2} \leq \lambda \sqrt{|S||T|}$$

This suffices for the proof. □

3 Edge Expansion

Theorem 11. *For any subset S of the set of vertices with $|S| \leq \frac{n}{2}$*

$$e(S, S^c) \geq \left(\frac{d - \lambda}{2} \right) |S|$$

Proof. By expander mixing lemma

$$\begin{aligned} e(S, S^c) &\geq \frac{d}{n}|S||S^c| - \lambda \sqrt{|S||S^c| \left(1 - \frac{|S|}{n}\right)\left(1 - \frac{|S^c|}{n}\right)} \\ &= \frac{d}{n}|S||S^c| - \frac{\lambda}{n}|S||S^c| \\ &= \frac{n - |S|}{n} \cdot (d - \lambda)|S| \end{aligned}$$

If $|S| \leq \alpha n$ then $\frac{n-|S|}{n} \geq (1-\alpha)$ and we have

$$e(S, S^c) \geq (1-\alpha)(d-\lambda)|S|$$

We have proved a generalization of this theorem, which is the particular case with $\alpha = 1/2$. $\square$

4 Vertex Expansion

For every subset S of the vertices with $|S| \leq \alpha n$

$$\begin{aligned}
|\text{supp}(A\mathbb{1}_S)| &\geq \frac{||A\mathbb{1}_S||_1^2}{||A\mathbb{1}_S||_2^2} \\
&= \frac{\sum a_i^2 \lambda_i^2}{d|S|} \\
&\geq \frac{d^2|S|^2}{\frac{d^2|S|^2}{n} + \lambda^2|S|} \\
&= \frac{|S|}{\frac{|S|}{n} + \frac{\lambda^2}{d^2}} \\
&\geq \left(\alpha + \frac{\lambda^2}{d^2}\right)^{-1} |S|
\end{aligned}$$

5 Error Correction Code Based on Spectral Expanders and Tanner Codes

Consider a d-regular graph G, we label the edges with zeros and ones (n vertices and $m = (n-d)/2$ edges). $(y_e)_{e \in E} \in \mathbb{F}_2^m$. Take code $C_0 \subset \mathbb{F}_2^d$ distance $\delta_0 d$. Ask that for all $v \in V$ $(y_e)_{v \in e} \in C_0$.

$$C = \{y \in \mathbb{F}_2^n : y = (y_e)_{e \in E} \text{ such that for all } v \in V \ (y_e)_{v \in e} \in C_0\}$$

Can there be a code word with very few 1s? Let $y \in C$ be a nonzero code. Let $F \subset E$ be the set of edges with $y = 1$. Let $S \subset V$ be the set of vertices touching some edge of F. For each $v \in S$ we find v touches $\geq \delta_0 d$ edges of F.

Claim. $|F| \geq \Omega(\delta_0^2) \cdot m$

Proof. Since $e(S, S) \geq \delta_0 d|S|$ and

$$\left| e(S, S) - \frac{|S|^2 d}{n} \right| \leq \lambda |S|$$

then

$$\delta_0 d|S| \leq e(S,S) \leq |S| \left(\frac{|S|d}{n} + \lambda \right)$$

$$\delta_0 d \leq \frac{|S|d}{n} + \lambda$$

$$|S| \geq n \left(\delta_0 - \frac{\lambda}{d} \right)$$

$$|F| \geq \frac{d\delta_0}{2}|S| = \frac{dn}{2}(\delta_0^2 - \frac{\lambda}{d}\delta_0)$$

The conclusion follows in the limit as $d \to \infty$. □