What does it mean for a graph to be vertex-transitive?

The graph $G = (V, E)$ is vertex transitive if for every pair of vertices $v, w \in V$, there is an automorphism of $G$ sending $v$ to $w$.

Let $G$ be a simple vertex-transitive graph. Assume that $G$ is a union of two disjoint connected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. That is:

$$G = G_1 \cup G_2 \quad \text{and} \quad G_1 \cap G_2 = \emptyset.$$  

Show that $|V_1| = |V_2|$.

Choose a vertex $x \in V_1$ and $y \in V_2$. Since $G$ is vertex transitive, there is an automorphism, given by the bijection

$$\theta: V \to V, \quad \text{and} \quad \eta: E \to E$$

such that $\theta(x) = y$. Let $z$ be any other vertex in $V_1$. Since $G_1$ is connected, there is path

$$x = v_1, e_1, v_2 \ldots e_{k-1}, v_k = z$$

connecting $x$ to $z$. The image of this path

$$y = \theta(x) = \theta(v_1), \eta(e_1), \theta(v_2) \ldots \eta(e_{k-1}), \theta(v_k) = \theta(z)$$

connects $y = \theta(x)$ to $\theta(z)$. But there is no edge between $G_1$ and $G_2$. Hence this path is contained in $G_2$ and, in particular, $z \in G_2$. This prove that

$$\theta(V_1) \subset V_2 \implies |V_1| \leq |V_2|.$$  

Similar, we can show $\theta^{-1}(V_2) \subset V_1$ and hence $|V_2| \leq |V_1|$. Therefore, $|V_1| = |V_2|$. 

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