• Define the cartesian product between two simple graphs $G$ and $H$.

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$. Then, $G \Box H$ is the graph whose vertex set is $V_G \times V_H$ so that, two vertices $(v_1, w_1), (v_2, w_2) \in V_G \times V_H$ are connected if

$$(v_1 = v_2 \text{ and } w_1w_2 \in E_H) \text{ or } (w_1 = w_2 \text{ and } v_1v_2 \in E_G)$$

• For simple graphs $G$ and $H$, show that if $G$ and $H$ are vertex transitive, so is $G \Box H$.

Let $(v_1, w_1), (v_2, w_2)$ be two vertices in $G \Box H$. We need to find an automorphism of $G \Box H$ sending one to the other.

Since $G$ is vertex transitive, there is an automorphism of $G$, represented by the bijection

$$\alpha: V_G \to V_G,$$

so that $\alpha(v_1) = v_2$. Similarly, since $H$ is vertex transitive, there is an automorphism of $H$, represented by the bijection

$$\beta: V_H \to V_H,$$

so that $\beta(w_1) = w_2$. Then,

$$\theta: V_G \times V_H \to V_G \times V_H, \text{ defined by } \theta(v, w) = (\alpha(v), \beta(w))$$

is also a bijection and $\theta(v_1, w_1) = (v_2, w_2)$.

We need to show that $\theta$ induces an automorphism, of $G \Box H$. For two vertices $(v_1, w_1)$ and $(v_2, w_2)$ in $G \Box H$, we have

$$(v_1, w_1) \text{ is connected to } (v_2, w_2)$$

$$\iff \begin{bmatrix} v_1 = v_2 \text{ and } w_1w_2 \in E_H \\ w_1 = w_2 \text{ and } v_1v_2 \in E_G \end{bmatrix}$$

$$\iff \begin{bmatrix} \alpha(v_1) = \alpha(v_2) \text{ and } \beta(w_1)\beta(w_2) \in E_H \\ \beta(w_1) = \beta(w_2) \text{ and } \alpha(v_1)\alpha(v_2) \in E_G \end{bmatrix}$$

$$\iff \theta(v_1, w_1) \text{ is connected to } \theta(v_2, w_2)$$

This finishes the proof.