To receive full credit you must show all your work. If you run out of room for an answer, continue on the back of the page.

This exam has 4 questions, for a total of 40 points.

1. (10 points) Define the following terms and expressions:

(a) Partial ordering.

A partial ordering on a set \( X \) is a relation that is transitive, irreflexive and anti-symmetric.

(b) Tournament.

A tournament is a directed graph \( G \) such that, for every two vertices \( v \) and \( w \), either \((v, w)\) or \((w, v)\) is an arc in \( G \) (not both).

(c) Degree sequence.

The degree sequence of a graph \( G \) is the sequence of integers

\[
d(v_1) \leq d(v_2) \leq \ldots \leq d(v_n).
\]

where \( v_i \) are vertices of \( G \) and \( d(v_i) \) is the degree of \( v_i \) in \( G \).

(d) Spanning sub-tree.

The spanning subtree of a graph \( G \) is a subgraph \( T \subset G \) that is a tree and has the same vertex set as \( G \).

(e) Hamiltonian.

A hamiltonian for a graph \( G \) is a spanning subgraph of \( G \) that is a cycle.
2. (10 points) Let $G$ be the Peterson graph depicted below.

(a) Is $G$ acyclic? (Explain with an argument or an example.)

No. The graph $(w_1 - w_2 - w_3 - w_4 - w_5 - w_1)$ is a cycle.

(b) Does $G$ have a Hamilton Path? (Explain with an argument or an example.)

Yes. Namely $(w_1 - w_2 - w_3 - w_4 - w_5 - v_5 - v_2 - v_4 - v_1 - v_3)$.

(c) Give an explicit automorphism of $G$ sending $v_1$ to $w_2$. You do not need to verify that the map you write is in fact an automorphism.

Here is one:

$v_1 \rightarrow w_2, \quad v_2 \rightarrow v_4, \quad v_3 \rightarrow w_3, \quad v_4 \rightarrow v_2, \quad v_5 \rightarrow w_4, \quad w_1 \rightarrow w_1, \quad w_2 \rightarrow v_1, \quad w_3 \rightarrow w_3, \quad w_4 \rightarrow v_5, \quad w_5 \rightarrow w_5$.

(Bonus) Is there an automorphism of $G$ sending $v_1 w_1$ to $w_1 w_2$?

Yes, the above map works here as well.
3. (10 points) Show that if a graph $G$ has no odd cycles then it is bipartite.

It is enough to show the problem for each connected component. Hence, without loss of generality, we can assume $G$ is connected.

Let $v$ be any vertex in $G$. Define $X$ be the set of vertices $x$ in $G$ so that the shortest path from $v$ to $x$ has an even length. Let $Y$ be the complement of $X$, namely, the set of vertices $y$ so that the shortest path from $v$ to $y$ has an odd length.

Then there is no edge between vertices $x, x' \in X$, otherwise, the union of paths $[v, x], [v, x']$ and the edge $xx'$ is an odd cycle in $G$. Similarly, there is no edge between vertices $y, y' \in Y$ because otherwise, the union of paths $[v, y], [v, y']$ and the edge $yy'$ is an odd cycle in $G$. Therefore, $G$ is bipartite.

(Bonus) Show that if $G$ is triangle free and $\delta > \frac{2n}{5}$ then $G$ is bipartite.

We show by induction that $G$ has no odd cycles. We know $G$ has no 3–cycles. Assume for contradiction that there is a 5–cycle $v_1 - v_2 - v_3 - v_4 - v_5$. Let $V_i$ be the set of vertices adjacent to $v_i$. Then

$$\sum_{i=1}^{5} |V_i| > 5 \times \frac{2}{5}n = 2n.$$ 

Hence, by the pigeonhole principle, one vertex $x$ is contained in at least 3 of them. That is, $x$ is connected to two adjacent vertices, which means there is a triangle. This is a contradiction.

In general, assume for contradiction that there is a $(2k+1)$–cycle $v_1 - \ldots - v_{2k+1}$ $(k > 2)$ and no shorter odd cycles. As above, let $V_i$ be the set of points adjacent to $v_i$. Then

$$\sum_{i=1}^{2k+1} |V_i| > (2k + 1) \times \frac{2}{5}n > 2n.$$ 

Hence, by the pigeonhole principle, one vertex $x$ is contained in at least 3 of them, namely $v_p, v_q, v_r$. If any of these are adjacent, then we have a triangle. Otherwise, there a pair (say $v_i, v_j$) where $i - j$ is odd and $|i - j| \leq (2k-2)$. Then, subpath path $[v_p, v_q]$ in the the $(2k+1)$–cycle union the edges $v_px, xv_q$ is an odd cycle of smaller length. That is contradiction.

Since there are no odd cycles, by the first part of the problem, $G$ is bipartite.
4. (10 points) Let $G[X,Y]$ be a simple 4–regular bipartite graph with 12 vertices. Show that $G$ contains a 4–cycle.

Since $G$ is regular, $X$ and $Y$ have the same size. That is each set has exactly 6 vertices. Pick $x_1, x_2 \in X$ and let $V_1$ and $V_2$ be the set of vertices they connect to respectively. Note that $V_1$ and $V_2$ are subsets of $Y$ and have 4 elements each. But $Y$ has 6 elements. Hence $V_1 \cap V_2$ has at least 2 elements, $y_1, y_2$. Then

$$x_1 - y_1, x_2 - y_2 - x_1$$

is a 4-cycle.