1. Exercises from 5.7/5.8

**Theorem 1.** Let $S$ be an oriented surface in $\mathbb{R}^3$ which is bounded by a piecewise smooth curve $\partial S$, where the boundary is given an orientation consistent with $S$, then if $\mathbf{F}(x, y, z)$ is a $C^1$ vector field defined on a neighbourhood of $S$ in $\mathbb{R}^3$:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$$

**Problem 1.** (Folland 5.7.2) Evaluate:

$$\int_C y \, dx + y^2 \, dy + (x + 2z) \, dz$$

where $C$ is the curve given by the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $y + z = a$, oriented counterclockwise as viewed from above.

- We will apply Stokes’ theorem.
- First compute the gradient of $\mathbf{F}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & y^2 & x + 2z \end{vmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

- Notice that $\nabla \times \mathbf{F}$ is collinear with the normal of the plane $y + z = a$, so pick $S$ to be the region of the plane bounded by $C$.
- This choice gives $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = -2$, so the integral over the whole surface is just $-2 \cdot \text{(Area of the region of the plane inside } C)$.
- It is clear that the plane $x + y = a$ cuts the sphere $x^2 + y^2 + z^2 = a^2$ in a circle (this was worked out in a previous tutorial), and the radius of this circle is $a/\sqrt{2}$ (draw the picture).
- Now, by Stokes’ theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = -2(\pi a^2/2) = -\pi a^2$$

- Caveat: The final answer is different from the solution Folland has given in the answer key, so there may be a mistake somewhere.

**Problem 2.** (Folland 5.8.1(ef)) Determine whether the following vector fields is the gradient of a function $f$, and if so, find $f$.

1. $\mathbf{F}_1(x, y, z) = (y - z)\mathbf{i} + (x - z)\mathbf{j} + (x - y)\mathbf{k}$
2. $\mathbf{F}_2(x, y, z) = 2xy\mathbf{i} + (x^2 + \log z)\mathbf{j} + ((y + 2)/z)\mathbf{k}$

Recall that on an open convex set, there exists a $C^1$ function $f(x, y, z)$ such that $\mathbf{F}(x, y, z) = \nabla f$ if and only if $\nabla \times \mathbf{F} = 0$. Since $\mathbb{R}^3$ is a convex set, we just apply the theorem. For the first vector field,

$$\nabla \times \mathbf{F}_1(x, y, z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y - z & x - z & x - y \end{vmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \neq 0$$

So the first vector field is not the gradient of a function. For the second vector field,

$$\nabla \times \mathbf{F}_2(x, y, z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2xy & x^2 + \log z & \frac{y + 2}{z} \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So there exists $f(x, y, z)$ such that $\nabla f = \mathbf{F}_2$. Let’s find $f$. We have to solve the three equations:

$$\partial_x f = 2xy$$
\[ \partial_y f = x^2 + \log z \]
\[ \partial_z f = \frac{y + 2}{z} \]

Integrating the first equation with respect to \( x \) gives \( f(x, y, z) = x^2y + g(y, z) \). Differentiating with respect to \( y \) gives:
\[ x^2 + \log z = \partial_y f = x^2 + \partial_y g \Rightarrow \partial_y g = \log z \Rightarrow g(y, z) = y \log z + h(z) \]

So \( f(x, y, z) = x^2y + y \log z + h(z) \). Differentiating with respect to \( z \) gives:
\[ \frac{y + 2}{z} = \frac{y}{z} + h'(z) \Rightarrow h'(z) = \frac{2}{z} \Rightarrow h(z) = 2 \log z + C \]

This gives a final solution:
\[ f(x, y, z) = x^2y + (y + 2) \log z + C \]

**Problem 3.** Determine whether the following vector fields are the curl of a vector field \( \mathbf{F} \), and if so, find \( \mathbf{F} \).

- \( \mathbf{G}_1(x, y, z) = (x^3 + yz) \mathbf{i} + (y - 3x^2y) \mathbf{j} + 4y^2 \mathbf{k} \)
- \( \mathbf{G}_2(x, y, z) = (xy + z) \mathbf{i} + xz \mathbf{j} - (yz + x) \mathbf{k} \)

(Thm. 5.63) Recall that on an open convex set, \( \mathbf{G}(x, y, z) = \nabla \times \mathbf{F} \) for some vector field \( \mathbf{F} \) if and only if \( \nabla \cdot \mathbf{G} = 0 \). So, by the theorem, all we have to do is compute the divergence of the vector fields.

\[ \nabla \cdot \mathbf{G}_1 = 3x^2 + 1 - 3x^2 + 0 = 1 \neq 0 \]
Therefore \( \mathbf{G}_1 \) is not the curl of a vector field. On the other hand,
\[ \nabla \cdot \mathbf{G}_2 = y + 0 - y = 0 \]
So \( \mathbf{G}_2 \) is the curl of some vector field, \( \mathbf{F} \). Let’s find \( \mathbf{F} \). We may assume that \( F_3 = 0 \), then:
\[ \nabla \times \mathbf{F} = \begin{pmatrix} -\partial_z F_2 \\ \partial_x F_1 \\ \partial_x F_2 - \partial_y F_1 \end{pmatrix} = \mathbf{G}_2 \]
Now we can solve for some of the components:
\[ F_2(x, y, z) = -xyz - z^2/2 + \varphi(x, y) \]
\[ F_1(x, y, z) = xz^2/2 - \psi(x, y) \]
So combining these two results we get:
\[ -yz - x = \partial_x F_2 - \partial_y F_1 = -yz + \partial_x \varphi + \partial_y \psi \]
We have the freedom now to choose \( \psi(x, y) = 0 \), then:
\[ \partial_x \varphi = -x \Rightarrow \varphi(x, y) = -x^2/2 \]
Then all together, one suitable vector field \( \mathbf{F} \) is given by:
\[ \mathbf{F}(x, y, z) = (xz^2/2) \mathbf{i} - (xyz + z^2/2 + x^2/2) \mathbf{j} \]