1. Exercises from 3.2

In this tutorial we’ll study different ways of representing smooth curves. There are three principle ways to do this. For curves in \( \mathbb{R}^2 \), they are as follows:

1. Explicitly, as the graph of a function \( y = f(x) \)
2. Explicitly by a parameterization \( t \mapsto (f_1(t), f_2(t)) \)
3. Implicitly, by the vanishing of a function \( S = \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 0\} \)

The implicit function theorem implies the local equivalence of these statements.

We say that a curve is smooth if every point has a neighbourhood on which the curve is the graph of a differentiable function. There are two obvious ways a curve can fail to be smooth: (1) It can intersect itself, or (2) it can have a cusp.

**Example of a smooth curve:** Let \( S = \{(x, y) \mid F(x, y) = y - x^2 = 0\} \). We can also think of \( S \) as the graph of the map \( f : x \mapsto x^2 \), or alternatively, as the image of the curve \( \gamma(t) : (-\infty, \infty) \to \mathbb{R}^2 \), \( \gamma(t) = (t, t^2) \). This curve is smooth almost by definition, since it is the map of \( y = f(x) = x^2 \), a differentiable map.

**Example of a non-smooth curve:** Let \( S = \{(x, y) \mid x^3 - y^2 = 0\} \), then we can define \( S \) piecewise as a curve by:

\[
\gamma(t) = (t^2, t^3) \quad t \in (-\infty, \infty)
\]

Alternatively, we can think of \( S \) as the graph of the function \( f(y) = y^{2/3} \). Notice though that this is not differentiable at the origin, since \( f'(y) = (2/3)y^{-1/3} \) is not defined at \( y = 0 \). This shows that \( S \) is not a smooth curve.

**Problem 1.** Let \( F(x, y) = xy(x+y-1) \), and set \( S = \{(x, y) \mid F(x, y) = 0\} \). Sketch \( S \). Is \( S \) smooth?

Near which points is \( S \) the graph of a function \( y = f(x) \), or \( x = f(y) \)?

- \( F(x, y) = 0 \) if and only if \( x = 0 \), or \( y = 0 \), or \( y = 1 - x \).
- (Draw \( S \)).
- Thm. 3.11 says that if \( a \in S \) and \( \nabla F(a) \neq 0 \), then \( S \) is the graph of a \( C^1 \) function in a neighbourhood of \( a \). Taking the contrapositive, if we want to find possible points where the curve \( S \) is not smooth, then we should look for points in \( S \) such that \( \nabla F(a) = 0 \).

\[
\nabla F = \begin{pmatrix}
y(2x + y - 1 + xy) \\
x(2y + x - 1 + xy)
\end{pmatrix}
\]

- Case 1: \( x = 0 \) and \( y = 0 \).
- Case 2: \( y = 0 \) and \( x \neq 0 \), then \( 2y + x - 1 + xy = x - 1 = 0 \) implies \( x = 1 \).
- Case 3: \( x = 0 \) and \( y \neq 0 \), then \( 2x + y - 1 + xy = y - 1 = 0 \) implies \( y = 1 \).
- Case 4: \( x \neq 0 \) and \( y \neq 0 \), then \( 2y + x - 1 + xy = 0 \) and \( 2x + y - 1 + xy = 0 \). Subtracting the second from the first gives \( y = x \). Now we need \( x^2 + 3x - 1 = 0 \), which can be solved to give \( y_0 = x_0 = (-3 \pm \sqrt{13})/2 \). However, \( F(x_0, y_0) \neq 0 \) so this point is not in \( S \).
- Near each of the points where \( \nabla F = 0 \), \( S \) is a union of two lines; therefore \( S \) could not be the graph of a single-valued function near any of these points.
- We have found the points of \( S \) such that \( \nabla F = 0 \), so by thm. 3.11 we know that \( S \) can be represented by the graph of a function near every point except \((0, 0), (0, 1), \) and \((1, 0)\).

**Problem 2.** Let \( \gamma(t) = (t^3 - 1, t^3 + 1) \). Is \( \gamma(t) \) a smooth curve? Sketch the curve. Examine \( S \) near any points where \( \gamma'(t) = 0 \).

- If we take \( x = \gamma_1(t), y = \gamma_2(t) \), then \( x - y + 2 = 0 \).
- Define \( F(x, y) = x - y + 2 \), then \( \nabla F(x, y) = (1, -1) \neq 0 \) so the curve \( \gamma(t) \) must be smooth
• (Sketch the plane)
• Notice that $\gamma'(t) = (3t^2, 3t^2)$ which has a zero at $t = 0$, however, the curve is still smooth at the point $(-1, 1)$.