Let $X$ be a topological space.
Let $S \subseteq X$

The **relative topology** on $S$ (as a subset of $X$) is the topology where a subset $U$ of $S$ is open if there exists $V$, an open subset of $X$ such that $U = V \cap S$

**Eg.** $X = \mathbb{R}$ usual topology
$S = [0,1]$.  

$U = (3/4, 1]$. (is an open subset on relative topology but not open in $\mathbb{R}$)

Recall: If $f: X \rightarrow Y$ is continuous, if $f^{-1}(U)$ is open (in $X$) whenever $U$ is open (in $Y$).

**Definition:** The topology $X$ is disconnected if there exists $U,V$ open subsets of $X$ such that $X = U \cup V$ 

& $U \cap V = \emptyset$ (and neither $U$ nor $V$ is $\emptyset$), If there is no such disconnection, then $X$ is connected.

**Theorem:** If $X$ is connected topological space, and $f:X \rightarrow Y$ is continuous, then $f(X)$ is a connected topological space in the relative topology as a subset of $Y$

Proof: Suppose $f(x)$ was disconnected.
Then $f(x) = U$, Union $U_2$, each $U_i$ is open subset of $f(x)$
$U_1 \cap U_2 = \emptyset$, $U_1 \neq 0$, $U_2 \neq 0$.
There exists $V_1,V_2$ open subsets of $Y$ such that
$U_i = V_1 \cap f(X) \text{ } & \text{ } U_2 = V_2 \cap f(X)$

Let $W_1 = f^{-1}(V_1)$ & $W_2 = f^{-1}(V_2)$

$f$ continuous $\Rightarrow W_1,W_2$ open (definition – inverses are open)

If $x \in W_1 \cap W_2$, then $f(x) \in V_1 \cap V_2$. But also $f(x) \in f(X)$
$\therefore f(x) \in V_1 \cap V_2 \cap f(x) = (V_1 \cap f(x)) \cap (V_2 \cap f(x)) = U_1 \cap U_2$

But $U_1 \cap U_2 \neq 0$, so no such $x$.
$\therefore W_1 \cap W_2 = \emptyset$, clearly $W_1 \cup W_2 = X$. 
U_1 \neq 0 \Rightarrow W_1 \neq \emptyset, U_2 \neq 0 \Rightarrow W_2 \neq \emptyset
\therefore W_1 \cup W_2 is a disconnection of X. Contradiction.

**Corollary:** If \( f: [a,b] \to \mathbb{R} \), then \( f \) assumes all values between \( f(a) \) & \( f(b) \) (Intermediate Value Theorem)

Proof: \([a,b]\) is connected (in its relative topology) – this really requires proof.
By above theorem, \( f([a,b]) \) (image) is connected.
Suppose \( f(a) < f(b) \) (Etc. if other way as usual).

Suppose \( f(a) < k \leq f(b) \) & no \( x \in [a,b] \) satisfies \( f(x) = k \).

Must show: not possible.

Let \( U_1 = \{ y: y < k \} \) \( U_2 = \{ y: y > k \} \)
\( U_1 \) & \( U_2 \) open, \( f \) continuous \( \Rightarrow f^{-1}(U_1) \) & \( f^{-1}(U_2) \) are open.

Claim: \( f^{-1}(U_1) \) & \( f^{-1}(U_2) \) is a disconnection of \([a,b]\).
Since \( f(x) \neq k \) \( \forall x \in [a,b] = f^{-1}(U_1) \cup f^{-1}(U_2) \)
\( f^{-1}(U_1) \neq \emptyset \) \( (a \in f^{-1}(U_1)) \), \( f^{-1}(U_2) \neq \emptyset \) \( (b \in f^{-1}(U_2)) \)
\( f^{-1}(U_1) \cap f^{-1}(U_2) = \emptyset \).

Contradicts \([a,b]\) being connected.

**Homeomorphic Examples**
\(\square\) \(\square\) – yes
\(\square\) \(\square\) – no because connectedness is not preserved.

\(\square\) \(\square\) No it is not, if you take a point out of ___, and get ___, this is disconnected but the circle will still be connected.

**Definition:** A component of a topological space is a connected subset of the space that is not properly contained in any other connected subset (ie., a maximal connected subset of the space).

**Theorem:** If each of \( S_\alpha \) is a connected subset of a topological space \( X \), and if \( \exists x \in \cap S_\alpha \),
then \( U S_\alpha \) is connected.

Proof: If \( U S_\alpha \) is disconnected, \( U S_\alpha = U_1 \cup U_2 \), each \( U_i \) open, non empty, \( U_1 \cap U_2 = \emptyset \) (disjoint).

There exists \( V_1, V_2 \) open in \( X \) with \( U_1 = V_1 \cap \left( U S_\alpha \right) \)
\( U_2 = V_2 \cap \left( U S_\alpha \right) \)

\( V_1 \cap S_2 \) \( V_2 \cap S_2 \)
\( x \in U_1 \cup U_2 \). Suppose \( x \in U_1 \) (etc if \( x \in U_2 \))
there exists \( \alpha_0 \) such that \( S_{\alpha_0} \cap U_2 \neq \emptyset \).

Then \( (V_1 \cap \left( U S_{\alpha_0} \right) \cup (V_2 \cap \left( U S_{\alpha_0} \right)) \) is a disconnection of \( S_{\alpha_0} \)

**Definition:** A collection \( \{ U_\alpha \} \) of sets covers a subset \( S \) if \( S \subset U_\alpha \). A subcovering of a covering
\{U_\alpha\} is a subcollection of the U_\alpha 's which still covers S.