(1) Show that there is an isomorphism of Lie groups \( \mathbb{R}^\times \to \mathbb{R} \times \mathbb{Z}/2 \) (where \( \mathbb{R}^\times \) is a group under multiplication and \( \mathbb{R} \) is a group under addition). Use this fact to show that \( \mathbb{R}^\times \) has two different structure of a real algebraic group. Show that these two different structures are non-isomorphic by showing that their complexifications are different.

(2) (a) Show that every algebraic representation of \( GL_n(\mathbb{R}) \) on (finite-dimensional) complex vector space is complete reducible (i.e. is the direct sum of irreducible subrepresentations).

(b) Give an example of a non-algebraic representation of \( GL_n(\mathbb{R}) \) on a (finite-dimensional) complex vector space which is not complete reducible.

(3) Consider \( G = SO_{2n}(\mathbb{C}) \). Find the roots and coroots of \( G \) as well the \( \psi_\alpha : SL_2(\mathbb{C}) \to SO_{2n}(\mathbb{C}) \).

Here is a suggestion to help you get started. Recall that \( SO_{2n}(\mathbb{C}) \) is the automorphisms of \( \mathbb{C}^{2n} \) which preserve a non-degenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \). Choose a basis \( v_1, \ldots, v_n, v_{-1}, \ldots, v_n \) for \( \mathbb{C}^{2n} \) such that

\[
\langle v_i, v_j \rangle = \begin{cases} 
1, & \text{if } i = j \pm n \\
0, & \text{otherwise}
\end{cases}
\]

Then the maximal torus is given by those elements of \( SO_{2n}(\mathbb{C}) \) which are diagonal with respect to this basis.

(4) Consider the group \( GO_{2n}(\mathbb{C}) \) which is called the orthogonal similitude group. It consists of those automorphisms of \( \mathbb{C}^{2n} \) which preserve the bilinear form up to a scalar. In other words for each \( g \in G \), there exists a scalar \( a \in \mathbb{C}^\times \) such that \( \langle gv, gw \rangle = a \langle v, w \rangle \) for all \( v, w \in \mathbb{C}^{2n} \). Find the root datum of \( GO_{2n}(\mathbb{C}) \) and compare with \( SO_{2n}(\mathbb{C}) \).

(5) Show that \( \Lambda^2 \mathbb{C}^4 \) carries a natural non-degenerate symmetric bilinear form. Use this fact to define a 2-to-1 cover \( SL_4(\mathbb{C}) \to SO_6(\mathbb{C}) \). What does this map look like on the level of root data?