Family Name: ______________________________________________________________

Given Name: ______________________________________________________________

Student Number: __________________________________________________________

Tutorial (T0101/T0201): __________________________________________________

Please write your solutions in the spaces provided below. If you run out of room for an answer, the last two pages have been intentionally left blank for this purpose. Please clearly indicate where your solution continues if you use one of these blank pages.

If you are unsure about what a question is asking or what things you may assume without proof, please ask. It is never my intention for the questions to be unclear. The worst that can happen is that I tell you I cannot answer the question.

Please try to organize your solutions neatly.
Question 1. (30 points, 3 points per part)

For each of the following questions, provide a brief justification for your answer unless otherwise instructed.

(a) Is \((\mathbb{Z}, T_{\text{discrete}}) \simeq (\mathbb{Q}, T_{\text{discrete}})\)?

(b) Does first countability imply second countability? If so, give a brief proof. If not, state (ie. no need to justify) a counterexample.

(c) Let \(T_{\text{usual}}\) be the usual topology on \(\mathbb{R}\). Is \(\{U \times V : U, V \in T_{\text{usual}}\}\) a topology on \(\mathbb{R}^2\)?

(d) Is \(\mathbb{R}_{\text{usual}} \times \mathbb{R}_{\text{ray}}\) with the product topology regular?

(e) True or false: \(\{[a, b) : a, b \in \mathbb{Q}\}\) is a basis for the lower limit topology on \(\mathbb{R}\) (ie. the Sorgenfrey line). (No justification required.)
(f) Let $S$ be the unit sphere in $\mathbb{R}^{2018}$, with its subspace topology inherited from the usual topology. Is $S$ a $T_3$ space?

(g) Let $X = \{1, 2, 3, 4, 5\}$. Determine whether each of the following topologies on $X$ is regular:

$$T_1 = \{\emptyset, \{1, 2\}, \{3, 4, 5\}, X\}$$

$$T_2 = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, X\}$$

(h) True or false: $T_4$ a hereditary property. (No justification required.)

(i) Let $f : \mathbb{R}_{\text{Sorgenfrey}} \to \mathbb{R}_{\text{Sorgenfrey}}$ be the absolute value function (i.e. the function defined by $f(x) = |x|$). Is $f$ continuous?

(j) Does every countable topological space have the countable chain condition? If so, give a brief proof. If not, give a counterexample.
Question 2. (9 points)

Prove that \((\mathbb{R}_{\text{Sorgenfrey}})^2\) with its product topology (i.e. the Sorgenfrey square) is separable but not hereditarily separable.
Question 3. (19 points)

(a) (3 points) Name three topological invariants that are both hereditary and finitely productive. (No justifications required.)

(b) (6 points) Choose one of the properties you named in part (a), and prove that it is both hereditary and finitely productive.

(c) (5 points) Let \((X, \mathcal{T})\) and \((Y, \mathcal{U})\) be topological spaces. Prove that \(Y \cong \{x\} \times Y\) for every \(x \in X\) (where the space on the right has the subspace topology inherited from the product topology on \(X \times Y\)).

It is also true that \(X \cong X \times \{y\}\) for all \(y \in Y\). You do not have to prove this.

(d) (5 points) Let \(\phi\) be a topological invariant that is both hereditary and finitely productive. Show that \(X \times Y\) has \(\phi\) if and only if \(X\) and \(Y\) both have \(\phi\).
Question 4. (12 points)

Let \((X, \mathcal{T})\) be a topological space. Define the diagonal subset of \(X \times X\) as

\[
\Delta := \{(x, x) \in X \times X : x \in X\}.
\]

Show that \((X, \mathcal{T})\) is Hausdorff if and only if \(\Delta\) is a closed subset of \(X \times X\) with the product topology.
Question 5. (20 points)

In this question, you will work with a new topological property. This property is a substantial strengthening of second countability.

Definition. A topological space \((X, \mathcal{T})\) is called **third countable** if every basis that generates the topology is countable.

(a) (2 points) Show that every finite topological space is third countable.

(b) (2 points) Show that if \((X, \mathcal{T})\) is third countable, then \(\mathcal{T}\) is itself countable. (Don’t overthink this.)

(c) (5 points) Suppose \((X, \mathcal{T})\) is a topological space such that there is an infinite collection \(\mathcal{U} \subseteq \mathcal{T}\) of pairwise disjoint open sets. Show that \(\mathcal{T}\) is uncountable.
(d) (7 points) Show that if $X$ is infinite and $(X, \mathcal{T})$ is Hausdorff, then it contains an infinite collection of pairwise disjoint open sets.

(Hint: First deal with the case in which $(X, \mathcal{T})$ is discrete. Then assume $(X, \mathcal{T})$ is not discrete, start by fixing a point $x \in X$ such that $\{x\}$ is not open, and then build the collection of open sets one by one.)

(e) (4 points) The four previous results combine to characterise all Hausdorff third countable spaces. State and prove a theorem to this effect. (Your theorem should be of the form: “A Hausdorff space is third countable if and only if ...”.)
This page is intentionally left blank to provide you with extra space.
This page is intentionally left blank to provide you with extra space.