Please write your solutions in the spaces provided below. If you run out of room for an answer, the last two pages have been intentionally left blank for this purpose. Please clearly indicate where your solution continues if you use one of these blank pages.

If you are unsure about what a question is asking or what things you may assume without proof, please ask. It is never my intention for the questions to be unclear. The worst that can happen is that I tell you I cannot answer the question.

Please try to organize your solutions neatly.
Question 1. (30 points, 3 points per part)

For each of the following questions, provide a brief justification for your answer unless otherwise instructed.

(a) Is $(\mathbb{Z}, T_{\text{discrete}}) \simeq (\mathbb{Q}, T_{\text{discrete}})$?

Solution: Yes. Both sets are countable, and so there exists a bijection between them. Any bijection is a homeomorphism between discrete topologies.

(b) Does first countability imply second countability? If so, give a brief proof. If not, state (ie. no need to justify) a counterexample.

Solution: No.

The Sorgenfrey line is first countable (with $\{ [x, x + \frac{1}{n}) : n \in \mathbb{N} \}$ being a countable local basis at any $x$), but not second countable (as per BL 4.10).

Also, any uncountable discrete space is first countable but not second countable.

(c) Let $T_{\text{usual}}$ be the usual topology on $\mathbb{R}$. Is $\{ U \times V : U, V \in T_{\text{usual}} \}$ a topology on $\mathbb{R}^2$?

Solution: No, because this collection of sets is not closed under unions. Elements of this set are like “open rectangles”, and the union of two such rectangles is not itself a rectangle. (This is very easy to demonstrate with a picture.)

(d) Is $\mathbb{R}_{\text{usual}} \times \mathbb{R}_{\text{ray}}$ with the product topology regular?

Solution: No. Many examples will show this. For example, the open upper half plane $\mathbb{R} \times (0, \infty)$ is open in this topology, and so the closed bottom half plane $\mathbb{R} \times (-\infty, 0]$ is closed. But it’s easy to see that any open set in the product that contains this closed set contains all other points in the plane, and so this set can’t be separated from any point.

A simpler version of this same example is that the closed bottom half of the $y$-axis (the set $\{ (0, y) : y \leq 0 \}$) is closed in this product, but you can’t separate it from the point $(0, 1)$.

(e) True or false: $\{ [a, b) : a, b \in \mathbb{Q} \}$ is a basis for the lower limit topology on $\mathbb{R}$ (ie. the Sorgenfrey line). (No justification required.)

Solution: False.

(We know the Sorgenfrey line is not second countable, so this can’t be a basis. For a more direct proof, note that there’s no element of $\mathcal{B}_Q$ that contains $\pi$ and is a subset of the open set $[\pi, \pi + 1]$.)
(f) Let $S$ be the unit sphere in $\mathbb{R}^{2018}$, with its subspace topology inherited from the usual topology. Is $S$ a $T_3$ space?

Solution: Yes. We know $\mathbb{R}_{\text{usual}}$ is both $T_1$ and regular, so it is $T_3$. We also know that regularity and $T_1$-ness are finitely productive (meaning $\mathbb{R}^{2018}$ is $T_3$), and hereditary (so any subspace of $\mathbb{R}^{2018}$, and in particular $S$, is $T_3$).

(g) Let $X = \{1, 2, 3, 4, 5\}$. Determine whether each of the following topologies on $X$ is regular:

\[ T_1 = \{\emptyset, \{1, 2\}, \{3, 4, 5\}, X\} \]
\[ T_2 = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, X\} \]

Solution: $T_1$ is regular. The only non-trivial open sets are both also closed, so each closed set “separates itself” from any other points.

$T_2$ is not regular. The set $C = \{5\}$ is closed in this topology, but the only open set that contains it is $X$, which also contains every other point. So $C$ can’t be separated from any points.

(h) True or false: $T_4$ a hereditary property. (No justification required.)

Solution: False. We learned that $(\mathbb{R}_{\text{Sorgenfrey}})^2$ is normal (and it’s certainly $T_1$), but that the anti-diagonal is a non-normal (and therefore non-$T_4$) subspace.

(i) Let $f : \mathbb{R}_{\text{Sorgenfrey}} \to \mathbb{R}_{\text{Sorgenfrey}}$ be the absolute value function (ie. the function defined by $f(x) = |x|$). Is $f$ continuous?

Solution: No. For example, the preimage of the open set $[1, 2)$ under $f$ is the set $(-2, -1] \cup [1, 2)$, which is not open in the Sorgenfrey line (since no basic open set of the form $[a, b)$ that contains $-1$ is contained in this set).

(j) Does every countable topological space have the countable chain condition? If so, give a brief proof. If not, give a counterexample.

Solution: Yes. There are many ways to see this. Suppose $X$ is a countable space, and let $\mathcal{U}$ be a collection of pairwise disjoint non-empty open sets. Every set in $\mathcal{U}$ must contain at least one point of $X$, so we can choose one and call it $x_U$. The sets are disjoint, so no point can be in more than one set. In other words, the map $\mathcal{U} \to X$ given by $U \mapsto x_U$ is injective, meaning $\mathcal{U}$ must also be countable (since $X$ is countable).

Another acceptable argument here is to note that every countable space is clearly separable (since the whole space is dense in itself), and we’ve seen on the Big List that every separable space is ccc.
Question 2. (9 points)

Prove that \((\mathbb{R}_{\text{Sorgenfrey}})^2\) with its product topology (ie. the Sorgenfrey square) is separable but not hereditarily separable.

Solution: We know that \(\mathbb{Q}\) is a countable dense subset in \(\mathbb{R}_{\text{Sorgenfrey}}\), since every nonempty interval of real numbers contains a rational. It follows that \(\mathbb{Q}^2\) is dense in \(X = (\mathbb{R}_{\text{Sorgenfrey}})^2\), since for any basic open set \(U = [a, b) \times [c, d)\) in the product topology, we can find rationals \(q \in [a, b)\) and \(p \in [c, d)\), and therefore \((q, p) \in U\). \(\mathbb{Q}^2\) is countable since it’s a finite Cartesian product of countable sets, and therefore \(X\) is separable.

On the other hand, let \(A\) be the antidiagonal in \(X\). That is

\[
A = \{ (x, -x) \in \mathbb{R}^2 : x \in \mathbb{R} \}.
\]

For any point \(p = (x, -x) \in A\), we can define the basic open set \(U_p = [x, x+1) \times [-x, -x+1)\) of \(X\). Then \(U_p \cap A = \{p\}\). This is very much easier to see if you draw a picture.

Since we can do this for every \(p \in A\), it follows that the subspace topology on \(A\) is discrete. Since \(A\) is clearly uncountable \((x \mapsto (x, -x)\) is a bijection \(\mathbb{R} \to A\)), it follows that \(A\) is not separable.
Question 3. (19 points)

(a) (3 points) Name three topological invariants that are both hereditary and finitely productive. (No justifications required.)

(b) (6 points) Choose one of the properties you named in part (a), and prove that it is both hereditary and finitely productive.

(c) (5 points) Let \((X, \mathcal{T})\) and \((Y, \mathcal{U})\) be topological spaces. Prove that \(Y \cong \{x\} \times Y\) for every \(x \in X\) (where the space on the right has the subspace topology inherited from the product topology on \(X \times Y\)).

It is also true that \(X \cong X \times \{y\}\) for all \(y \in Y\). You do not have to prove this.

(d) (5 points) Let \(\phi\) be a topological invariant that is both hereditary and finitely productive. Show that \(X \times Y\) has \(\phi\) if and only if \(X\) and \(Y\) both have \(\phi\).

Solution:

(a) Here are some possible choices: \(T_0, T_1, T_2, regular, T_3, first~countable, second~countable, countable~and~finite\) (the underlying space being countable and finite, in other words).

(b) One of many proofs could go here. Probably the easiest choice would have been to prove it for finiteness (for which the proof is so obvious as to almost be a triviality) or countability (for which the two things you have to prove were proved as elementary facts about countable sets).

Here’s the proof for \(T_2\):

(Hereditary). Let \(X\) be a \(T_2\) space and \(Y \subseteq X\). We want to show that \(Y\) is \(T_2\). So fix distinct points \(x, y \in Y\). Then in particular \(x\) and \(y\) are distinct elements of \(X\), so since \(X\) is \(T_2\) there are disjoint open subsets \(U\) and \(V\) of \(X\) containing \(x\) and \(y\) respectively. But then \(U \cap Y\) and \(V \cap Y\) are open in the subspace topology on \(Y\), contain \(x\) and \(y\) respectively, and are still disjoint.

(Finitely productive). As usual, it suffices to show that the product of two \(T_2\) spaces is \(T_2\). So let \(X\) and \(Y\) be \(T_2\) spaces. We want to show that \(X \times Y\), with its product topology, is \(T_2\). So fix two distinct points \(p_1 = (x_1, y_1)\) and \(p_2 = (x_2, y_2)\). Since \(p_1 \neq p_2\), it must be that either \(x_1 \neq x_2\) or \(y_1 \neq y_2\) (or both, of course).

Without loss of generality, suppose \(x_1 \neq x_2\). Then since \(X\) is \(T_2\), there are disjoint open sets \(U_1\) and \(U_2\) in \(X\) containing \(x_1\) and \(x_2\), respectively. But then \(U_1 \times Y\) and \(U_2 \times Y\) are disjoint (basic) open subsets of \(X \times Y\) that contain \(p_1\) and \(p_2\), respectively, as required.
(c) Fix spaces $X$ and $Y$ as in the question, and fix a point $x \in X$. In order to show that $Y \simeq \{x\} \times Y$ we need to define a bijection between them and show it’s continuous and that its inverse is continuous. We’ll actually show that’s continuous and open.

We use what I think is fair to call the obvious map: $f : Y \rightarrow \{x\} \times Y$, given by $f(y) = (x, y)$. This map is clearly a bijection.

To see that it’s continuous, note that the subspace topology on $\{x\} \times Y$ is generated by the basis obtained by intersecting basic open subsets of $X \times Y$ with the subspace. A basic open set of the product is of the form $U \times V$ in the usual way, and so the intersection of such a set with our subspace, if it’s non-empty, will always be of the form $\{x\} \times V$, for some open $V \subseteq Y$. The preimage of such a set under $f$ is $V$, which is of course open in $Y$.

To see that $f$ is open, fix an open set $V \subseteq Y$. Clearly we have $f(V) = \{x\} \times V$, which is open in the subspace topology since for example it equals $(X \times V) \cap (\{x\} \times Y)$.

(d) Let $\phi$ be as in the question, and let $X$ and $Y$ be topological spaces.

$(\Rightarrow)$. Suppose first that $X \times Y$, with its product topology, has $\phi$. Fix a point $x \in X$. Then $\{x\} \times Y$, as a subspace of the product, has $\phi$ since $\phi$ is hereditary. By part (c), this subspace is homeomorphic to $Y$, and so $Y$ also has $\phi$ since $\phi$ is a topological invariant. An exactly analogous argument (that you wouldn’t have needed to repeat) will show that $X$ must also have $\phi$.

$(\Leftarrow)$. Suppose that $X$ and $Y$ both have $\phi$. Then we can immediately say that $X \times Y$ must have $\phi$ since $\phi$ is finitely productive.
Question 4. (12 points)

Let \((X, T)\) be a topological space. Define the diagonal subset of \(X \times X\) as
\[
\Delta := \{(x, x) \in X \times X : x \in X\}.
\]
Show that \((X, T)\) is Hausdorff if and only if \(\Delta\) is a closed subset of \(X \times X\) with the product topology.

Solution: There’s essentially only one way to do this, but several ways one could write out the argument. The key fact here is the following, which we state as a lemma for the sake of clarify, and from which both directions of the proof follow easily.

Lemma. Let \(A\) and \(B\) be subsets of \(X\). Then \(A \cap B = \emptyset\) if and only if \((A \times B) \cap \Delta = \emptyset\)

Proof. Given a point \(x \in X\), then \((x, x) \in A \times B\) if and only if \(x \in A\) and \(x \in B\) if and only if \(x \in A \cap B\). The result follows immediately. 

Now, on to the problem at hand.

\((\Rightarrow)\). Suppose that \((X, T)\) is Hausdorff. We will prove that \(\Delta\) is closed in \(X \times X\) by showing that \((X \times X) \setminus \Delta\) is open. We show this by picking a point \((x, y) \notin \Delta\) and finding an open set containing it that doesn’t intersect \(\Delta\).

Let \((x, y) \notin \Delta\). Then \(x\) and \(y\) are distinct elements of \(X\), and since \(X\) is Hausdorff there are disjoint open sets \(U\) and \(V\) containing \(x\) and \(y\) respectively. Then \((x, y) \in U \times V\) and \((U \times V) \cap \Delta = \emptyset\) by the lemma.

\((\Leftarrow)\). Suppose \(\Delta\) is closed in \(X \times X\), and let \(x\) and \(y\) be distinct elements of \(X\). Then \((x, y) \in (X \times X) \setminus \Delta\) which is an open set by assumption. Therefore there is a basic open set \(U \times V\) containing \((x, y)\) and disjoint from \(\Delta\). But then \(x \in U\), \(y \in V\), and \(U \cap V = \emptyset\) by the lemma.
Question 5. (20 points)

In this question, you will work with a new topological property. This property is a substantial strengthening of second countability.

**Definition.** A topological space \((X, \mathcal{T})\) is called **third countable** if *every* basis that generates the topology is countable.

(a) (2 points) Show that every finite topological space is third countable.

*Solution:* If \((X, \mathcal{T})\) is a topological space in which \(X\) is finite then \(\mathcal{T}\) must also be finite, since there are only finitely many subsets of a finite set. Any given basis that generates \(\mathcal{T}\) must in particular be a subset of \(\mathcal{T}\), and therefore must also be finite (and therefore countable).

(b) (2 points) Show that if \((X, \mathcal{T})\) is third countable, then \(\mathcal{T}\) is itself countable. (Don’t overthink this.)

*Solution:* \(\mathcal{T}\), the topology, is itself a basis on \(X\) that generates \(\mathcal{T}\). Therefore if *every* basis that generates \(\mathcal{T}\) is countable, \(\mathcal{T}\) must be countable.

(c) (5 points) Suppose \((X, \mathcal{T})\) is a topological space such that there is an infinite collection \(\mathcal{U} \subseteq \mathcal{T}\) of pairwise disjoint open sets. Show that \(\mathcal{T}\) is uncountable.

*Solution:* Let \(\mathcal{U}\) be such a collection, and let \(\mathcal{V} = \{U_1, U_2, U_3, \ldots\}\) be an enumeration of any countably infinite subcollection of non-empty elements of \(\mathcal{U}\). We define a function \(f : \mathcal{P}(\mathbb{N}) \to \mathcal{T}\) in the following way:

\[
f(A) = \bigcup_{n \in A} U_n.
\]

In other words, \(f\) maps a subset \(A\) of \(\mathbb{N}\) to the union of all the open sets in \(\mathcal{V}\) with indices in \(A\).

First of all, any union of the form on the right is open (being a union of open subsets of a topological space), so this is indeed a map into \(\mathcal{T}\). Since we know that \(\mathcal{P}(\mathbb{N})\) is uncountable, it remains only to show that \(f\) is injective.

Let \(A\) and \(B\) be two distinct elements of \(\mathcal{P}(\mathbb{N})\). Then there must be some natural number \(k\) that is in one of them but not the other, so without loss of generality let’s assume some \(k\) is an element of \(A \setminus B\). Then clearly \(U_k \subseteq f(A)\), but \(U_k \cap f(B) = \emptyset\), since \(f(B)\) is a union of sets that are all disjoint from \(U_k\). Since \(U_k \neq \emptyset\), \(f(A) \neq f(B)\), as required.
(d) (7 points) Show that if $X$ is infinite and $(X, \mathcal{T})$ is Hausdorff, then it contains an infinite collection of pairwise disjoint open sets.

(Hint: First deal with the case in which $(X, \mathcal{T})$ is discrete. Then assume $(X, \mathcal{T})$ is not discrete, start by fixing a point $x \in X$ such that $\{x\}$ is not open, and then build the collection of open sets one by one.)

Solution: We follow the hint. First, assume $X$ is infinite and discrete. Then $\mathcal{U} = \{\{x\} : x \in X\}$ is an infinite collection of non-empty pairwise disjoint open subsets of $X$.

Next, assume $X$ is infinite and not discrete, and fix a point $x \in X$ such that $\{x\}$ is not open. We’ll use this point to inductively construct an infinite collection $\{W_1, W_2, \ldots\}$ of pairwise disjoint open sets. This argument is much easier to come up with and understand in a picture, so draw one as you read it. I’m writing it out in much more detail than I expect anyone to have given.

Since $X$ is infinite, let $y_1$ be a point other than $x$. Since the space is Hausdorff there are disjoint open sets $U_1$ and $W_1$ containing $x$ and $y_1$, respectively.

$U_1 \neq \{x\}$ by assumption, so there is some point $y_2 \in U_1$ other than $x$. Again since $X$ is Hausdorff, there are disjoint open sets $U_2 \subseteq U_1$ and $V_2$, containing $x$ and $y_2$, respectively. (To get $U_2 \subseteq U_1$, simply find any set that separates $x$ from $y_2$ and intersect it with $U_1$.) We now have two disjoint, nonempty open sets: $W_1$, and $W_2 := V_2 \cap U_1$.

One more step of the argument to illustrate what’s happening: Again, $U_2 \neq \{x\}$ by assumption, so there is a point $y_3 \in U_2$ other than $x$. Since $X$ is Hausdorff there are disjoint open sets $U_3 \subseteq U_2$ and $V_3$ containing $x$ and $y_3$, respectively. (To get $U_3 \subseteq U_2$, simply find any set that separates $x$ from $y_3$ and intersect it with $U_2$.) Now the open set $W_3 := V_3 \cap U_2$ is non-empty (it contains $y_3$) and disjoint from $W_1$ and $W_2$ (since $V_2$ is disjoint from $U_2$ and $W_2 \subseteq V_2$).

Continuing inductively in this way, suppose $n$ is fixed and we have found points $y_k$, and open sets $U_k$, $V_k$, and $W_k$, for all $k$ up to $n$, such that:

- $y_k$ does not equal $x$, and $y_k \in V_k$
- $x \in U_k$ and the $U_k$’s are nested
- $U_k \cap V_k = \emptyset$
- $W_k := V_k \cap U_{k-1}$

We know that $U_n \neq \{x\}$, so there is some point $y_{n+1} \in U_n$ other than $x$. Since $X$ is Hausdorff there are disjoint open sets $U_{n+1} \subseteq U_n$ and $V_{n+1}$ containing $x$ and $y_{n+1}$, respectively. Then the set $W_{n+1} := V_{n+1} \cap U_n$ is open and disjoint from all the previous $W_k$’s.
Upon completing the induction, the collection \( \{ W_k : k \in \mathbb{N} \} \) is infinite, and consists of pairwise disjoint non-empty open sets, as required.

(e) (4 points) The four previous results combine to characterise all Hausdorff third countable spaces. State a prove a theorem to this effect. (Your theorem should be of the form: “A Hausdorff space is third countable if and only if ...”.)

**Solution:**

**Theorem.** A Hausdorff space is third countable if and only if it is finite.

**Proof.** Let \((X, T)\) be a Hausdorff space.

\((\Leftarrow)\). If \(X\) is finite, then it is third countable by part (a).

\((\Rightarrow)\), by contrapositive. If \(X\) is infinite, then by part (d) there is an infinite collection of pairwise disjoint open subsets of \(X\). By part (c) this implies that \(T\) is infinite. By the contrapositive of part (b) this in turn implies that the space is not third countable.  \(\Box\)