Things You Should Know

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1 Basic Set Theory

I will assume students are familiar with all of these terms and symbols. Please don’t hesitate to ask about anything that seems unfamiliar.

In the following, $A, B, X, Y$ are sets. $I$ is an index set, $\{ A_\alpha : \alpha \in I \}$ and $\{ B_\alpha : \alpha \in I \}$ are families of sets indexed by $I$, and $C$ is a collection of sets.

- Empty set: $\emptyset$, the set with no elements.
- Subset: $A \subseteq B$ means “$x \in A \Rightarrow x \in B$”
- Power set: $\mathcal{P}(X) := \{ A : A \subseteq X \}$
- Union: $A \cup B := \{ x : x \in A \text{ or } x \in B \}$
- Intersection: $A \cap B := \{ x : x \in A \text{ and } x \in B \}$
- Complement: If $A \subseteq X$, then $X \setminus A := \{ x : x \in X \text{ and } x \notin A \}$
- Indexed union: $\bigcup_{\alpha \in I} A_\alpha := \{ x : \exists \alpha \in I, x \in A_\alpha \}$
- Non-indexed union: $\bigcup C := \bigcup_{X \in C} X$.
- Indexed intersection: $\bigcap_{\alpha \in I} A_\alpha := \{ x : \forall \alpha \in I, x \in A_\alpha \}$
- Non-indexed intersection: $\bigcap C := \bigcap_{X \in C} X$.
- Cartesian product of two sets: $X \times Y := \{ (x, y) : x \in X, y \in Y \}$

2 Functions

In the following, let $f : X \rightarrow Y$ be a function.

- $X$ is the domain of $f$.
- $Y$ is the codomain of $f$.
- $f(X) = \{ f(x) : x \in X \} \subseteq Y$ is the range or image of $f$.
- $f$ is injective (or one-to-one) if $f(a) = f(b)$ implies $a = b$. 
• $f$ is **surjective** (or **onto**) if its range is its entire codomain.

• $f$ is **bijective** if it is both injective and a surjective.

• The composition of two injective functions is again injective.

• The composition of two surjective functions is again surjective.

• The composition of two bijective functions is again bijective.

• Given a subset $B \subseteq Y$, the **preimage** of $B$ is the set $f^{-1}(B) := \{ x \in X : f(x) \in B \}$.

• If $f$ is an injection with range $Y$, then its inverse function $f^{-1} : Y \to X$ is (1) a function; and (2) injective.

3 **DeMorgan’s Laws and other interactions**

The following two expressions are generalized versions of what are called De Morgan’s Laws. They describe how unions and intersections interact with complementation.

1. $X \setminus \left( \bigcup_{\alpha \in I} A_\alpha \right) = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$

2. $X \setminus \left( \bigcap_{\alpha \in I} A_\alpha \right) = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$

The following are elementary facts about how functions interact with operations on subsets of their domains, codomains and ranges. Throughout the following, let $f : X \to Y$ be a function, and let $A, B \subseteq X$ and $C, D \subseteq Y$.

• $A \subseteq B$ implies $f(A) \subseteq f(B)$

• $C \subseteq D$ implies $f^{-1}(C) \subseteq f^{-1}(D)$

• $f(A \cup B) = f(A) \cup f(B)$

• $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

• $f(A \cap B) \subseteq f(A) \cap f(B)$

• $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

• $f(A) \setminus f(B) \subseteq f(A \setminus B)$

• $f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D)$
\[ f(X \setminus f^{-1}(Y \setminus C)) \subseteq C \]
\[ A \subseteq f^{-1}(f(A)), \text{ (with equality if } f \text{ is injective)} \]
\[ f(f^{-1}(C)) \subseteq C, \text{ (with equality if } f \text{ is surjective)} \]
\[ f^{-1}(Y \setminus C) = X \setminus f^{-1}(C) \]

4 Countability

We will spend some time on this in class, but I do expect these words to be familiar to you.

**Definition 1.** A set \( A \) is said to be **countably infinite** if there exists a bijection \( f : \mathbb{N} \to A \). A set \( A \) is said to be **countable** if it is finite or countably infinite. If \( A \) is infinite but not countably infinite, \( A \) is said to be **uncountable**.

The following theorem gives some equivalent conditions for being countable:

**Theorem 2.** For an infinite set \( A \), the following are equivalent:

1. \( A \) is countable.
2. There is an injection \( f : A \to \mathbb{N} \).
3. There is a surjection \( g : \mathbb{N} \to A \).

**Fact:** The following sets are countable:

- \( \mathbb{N}, \mathbb{Z}, \mathbb{Q} \), the set of algebraic numbers.
- Any infinite subset of a countable set.
- The Cartesian product of two countable sets (and, inductively, the Cartesian product of a finite number of countable sets).
- The union of finitely many countable sets.
- The union of a countable collection of countable sets.
- The countable union of some countable sets and some finite sets.

**Fact:** The following sets are uncountable:

- \( \mathbb{R}, \mathbb{R} \setminus \mathbb{Q} \) (the irrational numbers), the set of non-algebraic numbers (i.e. the set of transcendental numbers), \( \mathbb{R}^n \).
• Any superset of an uncountable set.
• The power set of any infinite set (countable or otherwise), e.g. \( \mathcal{P}(\mathbb{N}) \).
• The set \( \mathbb{N}^\mathbb{N} \) of functions from \( \mathbb{N} \) to \( \mathbb{N} \).

The following are two very useful combinatorial facts

**Theorem 3** (Pigeonhole Principle). *Let \( X \) be an infinite set, and \( A \) a finite set. If \( c : X \to A \) is a function, then there is an \( a \in A \) such that \( c^{-1}(a) \) is infinite.*

**Theorem 4** (Uncountable Pigeonhole Principle). *Let \( X \) be an uncountable set. If \( c : X \to \mathbb{N} \) is a function, then there is an \( n \in \mathbb{N} \) such that \( c^{-1}(n) \) is uncountable.*

The latter theorem can be restated in plain English as "If you try to put uncountably many pigeons into countably many holes, then there is a hole with uncountably many pigeons”.

5 Selected basic facts about \( \mathbb{R} \)

First recall: \( \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \). (For us, \( 0 \notin \mathbb{N} \).)

**Fact**: Between any two distinct real numbers:

• There is a rational number.
• There are infinitely many rational numbers.
• There is an irrational number.
• There are infinitely many irrational numbers.

**Fact**: Here are some useful facts from calculus:

• \( \bigcup \frac{1}{n}, 1 \bigcup n \in \mathbb{N} = (0, 1] \).
• \( \bigcup [0, n] \bigcup n \in \mathbb{N} = [0, \infty) \).
• \( \sum_{n \in \mathbb{N}} 2^{-n} = 1 \).