A1-1999 Find polynomials $f(x), g(x),$ and $h(x),$ if they exist, such that for all $x$,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} 
-1 & \text{if } x < -1 \\
3x + 2 & \text{if } -1 \leq x \leq 0 \\
-2x + 2 & \text{if } x > 0.
\end{cases}$$

B1-2005 Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers $a$. (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to $\nu$.)

- Do there exist $n \times n$ matrices with $AB - BA = 1$ (identity)?
- Find the zeroes of the polynomial $x^4 - 6x^3 + 18x^2 - 30x + 25$, knowing that the sum of two of its roots is 4.

B1-2004 Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that $r$ is a rational number such that $P(r) = 0$.

Show that the $n$ numbers

$$c_n r, c_n r^2 + c_{n-1} r, c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \ldots, c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.

- Find all polynomials $P(x)$ such that $(x + 1)P(x) = (x - 10)P(x + 1)$.

A1-2001 Consider a set $S$ and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$.

Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

B1-2003 Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2 y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?
B2-1993 For nonnegative integers \( n \) and \( k \), define \( Q(n,k) \) to be the coefficient of \( x^k \) in the expansion of \((1 + x + x^2 + x^3)^n\). Prove that
\[
Q(n,k) = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-2j},
\]
where \( \binom{a}{b} \) is the standard binomial coefficient. (Reminder: For integers \( a \) and \( b \) with \( a \geq 0 \), \( \binom{a}{b} = \frac{a!}{b!(a-b)!} \) for \( 0 \leq b \leq a \), with \( \binom{a}{b} = 0 \) otherwise.)

A2-1999 Let \( p(x) \) be a polynomial that is nonnegative for all real \( x \). Prove that for some \( k \), there are polynomials \( f_1(x), \ldots, f_k(x) \) such that
\[
p(x) = \sum_{j=1}^{k} (f_j(x))^2.
\]

B2-1999 Let \( P(x) \) be a polynomial of degree \( n \) such that \( P(x) = Q(x)P''(x) \), where \( Q(x) \) is a quadratic polynomial and \( P''(x) \) is the second derivative of \( P(x) \). Show that if \( P(x) \) has at least two distinct roots then it must have \( n \) distinct roots.

B2-2001 Find all pairs of real numbers \((x,y)\) satisfying the system of equations
\[
\frac{1}{x} + \frac{1}{2y} = (x^2 + 3y^2)(3x^2 + y^2) \\
\frac{1}{x} - \frac{1}{2y} = 2(y^4 - x^4).
\]

A4-1997 Let \( G \) be a group with identity \( e \) and \( \phi : G \to G \) a function such that
\[
\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)
\]
whenever \( g_1g_2g_3 = e = h_1h_2h_3 \). Prove that there exists an element \( a \in G \) such that \( \psi(x) = a\phi(x) \) is a homomorphism (i.e. \( \psi(xy) = \psi(x)\psi(y) \) for all \( x, y \in G \)).

B–4 Let \( n \) be a positive integer. Find the number of pairs \( P, Q \) of polynomials with real coefficients such that
\[
(P(X))^2 + (Q(X))^2 = X^{2n} + 1
\]
and \( \deg P > \deg Q \).

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