MAT 1110 - Additional exercises

March 8, 2013

Ex. 1: \[ \text{Recall that a sequence } 1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1 \text{ of alg. gp. is exact if: } \]
\[ \begin{align*}
& (i) \text{ it's set-theoretically exact} \\
& (ii) 0 \rightarrow \text{Lie } K \xrightarrow{dy} \text{Lie } G \xrightarrow{dy} \text{Lie } H \rightarrow 0 \text{ is exact.} \\
\end{align*} \]
(a) Show that \( \psi \) is a closed immersion \( \iff \) \( \psi \) is an immersion and \( d\psi \) is surjective.
(b) \( \text{--- } \psi \) is separable \( \iff \) \( \psi \) is unipotent and \( d\psi \) is surjective.
(c) Deduce the sequence is exact \( \iff \)
\[ \begin{cases} 
(ii) & \psi \text{ is a closed immersion} \\
(iii) & \psi \text{ is separable}.
\end{cases} \]
(d) If \( \text{char } k = 0 \), show that \( (i) \Rightarrow (iii) \).

Ex. 2: If \( N \leq H \leq G \) are closed subgp's, and \( N \triangleleft G \), then the 
\( N \)-natural map \( H/N \rightarrow G/N \) is a closed immersion (so we can think of 
\( H/N \) as a closed subgroup of \( G/N \)) and we have a canonical iso. 
\[ (G/N)/(H/N) \approx G/H, \text{ of homog. G-spaces} \]

Ex. 3: Suppose \( N, H \leq G \) are closed subgp's such that \( H \) normalizes \( N \), 
show that \( HN \text{ is a closed subgroup of } G \) and that we have a 
canonical iso. \( HN/N \approx H/(HN) \) of alg. gp's. \( \text{Assume char } k = 0 \)
Find a counterexample when \( \text{char } k > 0 \).

Ex. 4: Suppose \( \psi : G \rightarrow H \) is a morphism of alg. gp's. If \( \text{char } k = 0 \) show that 
\( \psi \) induces an iso. \( G/\ker \psi \approx \text{im } \psi \). If \( \text{char } k > 0 \), find a counterexample 
\( \psi \) induces an iso.

Ex. 5: Suppose \( H \) is a closed subgp of an alg. gp \( G \). Show that \( \text{both } H \\
\text{and } G/H \text{ are connected, then } G \text{ is connected.} \) (Use e.g. Springer Ex. 6.5.9(1))

Ex. 6: Suppose \( \psi : G \rightarrow H \) is a morphism of alg. gp's. If \( H_1 \leq H_2 \leq H \) are 
\( \text{closed subgp's, show that we have a canonical iso. } \psi^{-1}(H_2)/\psi^{-1}(H_1) \approx H_2/H_1 \). 
(\text{Hint: show } \text{Lie } \psi^{-1}(H_i) = (d\psi)^{-1} \text{Lie } H_i). \)