Path Homotopy

\( \alpha, \beta : I \to X \) (cts) paths of some end points.

\( \alpha \simeq_p \beta \) if \( F : I \times I \to X \) such that

\[
\begin{align*}
\text{constant path} & \quad \epsilon_p(s) = p \text{ for all } s \\
\text{reverse path} & \quad \overline{\alpha}(s) = \alpha(1 - s) \\
\text{concatenation} & \quad (\alpha * \beta)(s) = \begin{cases} 
\alpha(2s) & s \in [0, 1/2] \\
\beta(2s - 1) & s \in [1/2, 1]
\end{cases}
\end{align*}
\]

Theorem 57

\( \alpha : \) path from \( p \) to \( q \) \\
\( \beta : \) path from \( q \) to \( r \) \\
\( \gamma : \) path from \( r \) to \( x \)

(i) \( \alpha * \epsilon_q \simeq_p \alpha \simeq_p \epsilon_p * \alpha \)

(ii) \( \alpha * \overline{\alpha} \simeq_p \epsilon_p, \overline{\alpha} * \alpha \simeq_p \epsilon_q \)

(iii) \( (\alpha * \beta) * \gamma \simeq_p \alpha * (\beta * \gamma) \)

Proof:

(i) last time

\[
\begin{align*}
F(s, t) = \begin{cases} 
\alpha(2s) & s \leq t/2 \\
\alpha(t) & t/2 \leq s \leq 1 - t/2 \\
\overline{\alpha}(2s - 1) & s \geq 1 - t/2
\end{cases}
\end{align*}
\]
Suppose Similarly, Consider $p$

Intuition: 

Goal: Find Similarly, $E$

\[ (i) \]

\[ (ii) \]

\[ (iii) \]

\[ F(s, t) = \begin{cases} 
\alpha \left( \frac{4s}{t+1} \right) & t \geq 4s - 1 \\
\beta(4s - t - 1) & 4s - 2 \leq t \leq 4s - 1 \\
\gamma \left( \frac{4s - 2 - t}{2-t} \right) & t \leq 4s - 2 
\end{cases} \]

**Proposition 58**

If $\alpha \simeq_p \alpha'$, $\beta \simeq_p \beta'$ and $\alpha(1) = \beta(0)$, then $\alpha \ast \beta \simeq_p \alpha' \ast \beta'$. Also, $\bar{\alpha} \simeq_p \bar{\alpha}'$

**Proof:**

Fix $p \in X$. Let $\Omega(X, p) := \{ \text{loops at } p \}$, i.e. paths $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = p$. Then $\epsilon_p \in \Omega(X, p)$ $\alpha, \beta \in \Omega(X, p) \Rightarrow \bar{\alpha}, \alpha \ast \beta \in \Omega(X, p)$.

**Definition** The fundamental group $\pi_1(X, p)$ is the set of path homotopy equivalence classes in $\Omega(X, p)$ i.e. $\pi_1(X, p) = \Omega(X, p)/ \simeq_p$

**Example** $X \subset \mathbb{R}^n$ convex subset, then $\pi_1(X, p)$ consists of one element only (see Ex. last time)

Write $[\alpha]$ for the equivalence class of a path $\alpha$. By prop 58, $[\alpha] = [\alpha'] \Rightarrow [\bar{\alpha}] = [\bar{\alpha'}]$. So can define $[\bar{\alpha}] := [\bar{\alpha}]$, as the right-hand side only depends on $[\alpha]$. Similarly, can define $[\alpha] \ast [\beta] := [\alpha \ast \beta]$ since the right-hand side only depends on $[\alpha]$ and $[\beta]$. (by prop 58)

From Theorem 57 we get

**Corollary 59**

Suppose $[\alpha], [\beta], [\gamma] \in \pi_1(X, p)$, then

(i) $[\alpha] \ast [\epsilon_p] = [\alpha] = [\epsilon_p] \ast [\alpha]$

(ii) $[\alpha] \ast [\bar{\alpha}] = [\epsilon_p] = [\bar{\alpha}] \ast [\alpha]$

(iii) $([\alpha] \ast [\beta]) \ast [\gamma] = [\alpha] \ast ([\beta] \ast [\gamma])$. 

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This means that \( \pi_1(X, p) \) together with the operation \( *: \pi_1 \times \pi_1 \to \pi_1 \)

(i): \( \epsilon_p \) identity element.

(ii) \([\alpha]^{-1}\) inverse of \([\alpha]\) (identity and inverse elements are in fact unique).

(iii) \( * \) is associative.

**Examples**

\((\mathbb{Z}, +)\) group: \( \text{id} = 0 \), inverse of \( n = -n \), associative.

Similarly, \((\mathbb{Q}, +), (\mathbb{R}, +), \ldots \)

\( D_6 = \) symmetries of equilateral triangle. (2 rotations, 3 reflections, identity)

Find group of the circle §53, 54

\( S^1 := \{ z \in \mathbb{C} : |z| = 1 \} \)

Goal: \( \pi_1(S^1, 1) \cong (\mathbb{Z}, +) \) – bijection that preserves the group operation.

Intuition: we can assign its winding number \( \in \mathbb{Z} \).

\[ \pi : \mathbb{R} \to S^1 \text{ (cts)} \]

\[ x \mapsto e^{2\pi i x} \]

\[ e^{2\pi i x} = 1, \quad e^{2\pi i y} = \cos(2\pi x) + i \sin(2\pi x) \leftrightarrow x \in \mathbb{Z} \]

\[ e^{2\pi i x} = e^{2\pi i y} \leftrightarrow x - y \in \mathbb{Z} \]

The Idea is can lift a loop at \( 1 \in S^1 \) to a path starting at \( 0 \in \mathbb{R} \). Endpoint \( \in \mathbb{Z} \)

Consider \( U_+ := S^1 \setminus \{-1\}, \quad U_- := S^1 \setminus \{1\} \), open subsets.

**Proposition 60**

\[ \pi^{-1}(U_+) = \bigcup_{n \in \mathbb{Z}} \left( n - \frac{1}{2}, n + \frac{1}{2} \right) \text{ (disjoint union)} \quad \text{and} \quad \pi|_{(n-1/2, n+1/2)} : (n - 1/2, n + 1/2) \to U_+ \text{ is a homeomorphism.} \]

Similarly, for \( U_- \). So for any point \( x \in S^1 \), there is a neighborhood (either \( U_+ \) or \( U_- \)) that has a simple preimage.

**Proposition 61** (path lifting)
Suppose \( \alpha : I \to S^1 \) is a path. Fix \( x \in \pi^{-1}(\alpha(0)) \). Then there is a unique path \( \tilde{\alpha} : I \to \mathbb{R} \) such that \( \tilde{\alpha}(0) = x \) and \( \pi \circ \tilde{\alpha} = \alpha \).