§51, 52 Homotopy and Fundamental Groups

Suggested textbook: Armstrong “Basic Topology”

Let $I := [0, 1] \subset \mathbb{R}$

**Definition** $f, g : X \rightarrow Y$ cts, then $f, g$ are *homotopic* (write $f \simeq g$) if $\exists$ $F : X \times I \rightarrow Y$ such that such that $F(x, 0) = f(x), F(x, 1) = g(x), \forall x \in X$. $F$ is called *homotopy* (between $f, g$)

**Example**

$f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ rotation by angle $\alpha$ (around 0), then $f \simeq \text{id}$ (identity map)

Homotopy at time $t$ : rotation by $t\alpha$.

Focus on $X = I$, i.e. consider paths $I \rightarrow Y$.

**Definition** $\alpha, \beta : I \rightarrow X$ (cts) are path homotopic (write $\alpha \simeq_p \beta$) if $\alpha, \beta$ have same endpoints $p, q$.

$\exists F : I \times I$ (time) $\rightarrow X$ such that $F(s, 0) = \alpha(s), F(s, 1) = \beta(s), F(0, t) = p, F(1, t) = q$.

**Examples**

1) $X = \mathbb{R}^2$, Fix $p, q \in X$. Any two paths from $p$ to $q$ are homotopic.

Fix $s$, idea : connect by straight lines.

Explicitly: $F(s, t) = (1 - t) \alpha(s) + t\beta(s)$, clearly continuous.

Similarly this works fro all convex subsets in $\mathbb{R}^n$

2) $X = \mathbb{R}^2 \setminus \{0\} \equiv \mathbb{C} \setminus \{0\}$. We’ll see that $\alpha(s) = e^{i\pi s}, \beta(s) = e^{-i\pi s}$ not path homotopic.
Proposition 56

(Path) homotopy is an equivalence relation.

Proof: do only for \( \simeq_p \)

\[ \alpha \simeq_p \alpha \text{ obvious : } F(s, t) = a(s), \alpha \simeq_p \beta \Rightarrow \beta \simeq_p \alpha : \text{ reverse time } t \mapsto 1 - t \]

\[ \alpha \simeq_p \beta, \beta \simeq_p \gamma \Rightarrow \alpha \simeq_p \gamma \]

\[ F(s, t) = \begin{cases} F_1(s, 2t) & 0 \leq t \leq 1/2 \\ F_2(s, 2t - 1) & 1/2 \leq t \leq 1 \end{cases} \]

By pasting lemma (book 18.3) shows \( F \) is continuous. \( \square \)

Constant path: given \( p \in X \), let \( \epsilon_p : I \rightarrow X \) for all \( s \)

\[ s \mapsto p \]

Reverse path: given path \( \alpha \), define \( \overline{\alpha} : I \rightarrow X \)

\[ s \mapsto \alpha(1 - s) \]

Concatenation: If \( \alpha, \beta : I \rightarrow X \) such that \( \alpha(1) = \beta(0) \), then can define

\[ \alpha \ast \beta : I \rightarrow X, \ s \mapsto \begin{cases} \alpha(2s) & 0 \leq s \leq 1/2 \\ \beta(2s - 1) & 1/2 \leq s \leq 1 \end{cases} \text{ (continuous by pasting lemma)} \]

Proof: \( (\alpha \ast \beta) \ast \gamma \neq \alpha \ast (\beta \ast \gamma) \)

Theorem 57

(i) \( \alpha \) path from \( p \) to \( q \). \( \alpha \not\equiv_p \alpha \) and \( \epsilon_p \ast \alpha \equiv_p \alpha \)

Proof: Only do \( \epsilon_p \ast \alpha \equiv_p \alpha \).
Let \( F(s, t) = \begin{cases} p & t \geq 2s \\ a\left(\frac{2s-1}{2-t}\right) & t \leq 2s \end{cases} \)