Recall  Urysohn’s Lemma: $X$ is $T_4$ then $\forall A, B$ disjoint closed subset, $\exists f : X \to [0, 1]$ cts such that $f|_A = 0$ and $f|_B = 1$.

Remark  This works for $[a, b]$ instead of $[0, 1]$ since $[0, 1] \equiv [a, b]$.  

Theorem 52 (b)

$X_\lambda$ is $T_3 \forall \lambda \Rightarrow \prod_{\lambda \in \Lambda} X_\lambda$ is $T_3$.

Proof:

Use criterion of Lemma 50 ($\exists V$ open such that $x \in V \subset \overline{V} \subset U$). Fix $x = (x_\lambda) \in \prod X_\lambda \Rightarrow x \in U$  

So $\exists$ basic open nbd, $\prod_{\lambda \in \Lambda} V_\lambda$ of $x$ contained in $U$. ($V_\lambda$ open in $X_\lambda$ and $V_\lambda = X_\lambda$ for all but finitely many $\lambda$.)  

Since $X_\lambda$ is $T_3 \exists W_\lambda$ such that $x_\lambda \in W_\lambda \subset \overline{W_\lambda} \subset V_\lambda$. Whenever $V_\lambda = X_\lambda$ take $W_\lambda := X_\lambda$  

Then $W := \prod W_\lambda$ is open in $\prod X_\lambda$.

Claim : $\overline{W} \subset \prod V_\lambda$  

For any $\mu \in \Lambda$, $p_\mu : \prod X_\lambda \to X_\mu$ is cts. $\Rightarrow p_\mu(W) \subset p_\mu(\overline{W})$ (by Theorem 21) = $\overline{W_\mu} \subset V_\mu$. □

§35 Tietze Extension Theorem

Proposition 53

$Y$ metric space, suppose $f_n : X \to Y$ cts ($n \geq 1$), and $f : X \to Y$. If $f_n \to f$ uniformly, then $f$ is continuous.

uniformly means : $\forall \varepsilon > 0, \exists n_0 \geq 1$ such that $d(f_n(x), f(x)) < \varepsilon, \forall x \in X, \forall n \geq n_0$.

Example

$f_n : X \to \mathbb{R}$ cts, $M_n \in \mathbb{R}_{\geq 0}$ such that $|f_n(x)| \leq M_n, \forall x$ and $\sum M_n$ converges $\Rightarrow \sum f_n$ converges uniformly and its continuous. (Weierstrass M test)

Theorem 54 (Tietze)

$X$ be $T_4$ space, $A \subset X$ closed, $f : A \to [a, b]$ cts, then $\exists \tilde{f} : X \to [a, b]$ cts such that $\tilde{f}|_A = f$.

Proof:

WLOG, $[a, b] = [-1, 1]$ (they are homeo)  

Idea: use successive approximation.

If $g : A \to [-1, 1]$ satisfies.
\[ g(x) = \begin{cases} 
 1/3 & \text{if } f(x) \geq 1/3 \\
 \in [-1/3, 1/3] & \text{if } f(x) \in [-1/3, 1/3] \\
 -1/3 & \text{if } f(x) \leq -1/3 
\end{cases} \]

Then \(|f(x) - g(x)| \leq \frac{2}{3} \forall x \in A \)

\[ C := f^{-1}(\left[ \frac{1}{3}, 1 \right]) \text{ closed in } A \rightarrow \text{ closed in } X \text{ (as } A \text{ is)} \]

\[ D := f^{-1}(\left[ -1, -\frac{1}{3} \right]) \text{ closed in } A \rightarrow \text{ closed in } X \text{ (as } A \text{ is)} \]

\[ \Rightarrow C, D \text{ is disjoint closed} \]

Urysohn’s Lemma \[ \Rightarrow \exists f_1 : X \rightarrow \left[ -\frac{1}{3}, \frac{1}{3} \right] \text{ cts such that } f_1 |_C \equiv \frac{1}{3}, f_1 |_D \equiv -\frac{1}{3} \]

Moreover, \(|f(x) - f_1(x)| \leq \frac{2}{3}, \forall x \in A \).

Hence, \( f - f_1 |_A : A \rightarrow \left[ -\frac{2}{3}, \frac{2}{3} \right] \text{ cts} \)

The same argument gives: \( \exists f_2 : X \rightarrow \left[ -\frac{2}{3}, \frac{2}{3} \right] \text{ cts such that } |f(x) - f_1(x) - f_2(x)| \leq \left( \frac{2}{3} \right)^2, \forall x \in A \)

\[ \exists f_n : X \rightarrow \left[ -\frac{1}{3}, \frac{1}{3} \right] \text{ cts such that } \sum_{n=1}^{\infty} f_n(x) \leq \left( \frac{2}{3} \right)^n \Rightarrow \sum_{n=1}^{\infty} f_n(x) \text{ is uniformly convergent, so } \tilde{f} \text{ cts.} \]

|\tilde{f}(x)| \leq |\sum_{n=1}^{\infty} f_n(x)| \leq \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n = 1.

Get \( \tilde{f} : X \rightarrow [-1, 1] \text{ cts. Let } n \rightarrow \infty \text{ in } \tilde{f}, \Rightarrow \tilde{f} |_A = f. \Box \)