Last Time

Theorem 44(v) states “complete+totally bounded” ⇔ compact. This generalizes Heine–Borel.

\[ X \subset \mathbb{R}^n, \quad X \text{ bounded in Euclidean metric } \iff \text{ totally bounded} \]
\[ X \text{ closed } \iff X \text{ compete, since } \mathbb{R}^n \text{ complete} \]

Any compact metric space is continuous image of Cantor set! (proof will be linked on course website)

Separation Axioms

\( T_1 \) is equivalent of "points are closed", \( T_2 = \) Hausdorff. \( T_2 \Rightarrow T_1 \).
\( T_2 \) does not imply \( T_1 \), example : finite complement topology.

Definition \( X \) is **regular** or \( T_3 \) if \( X \) is \( T_1 \) and for any closed subset \( C \subset X \) and any \( x \not\in C \), \( \exists \) disjoint open sets \( U \) containing \( x \) and, \( V \supset C \).

\[
\begin{array}{c}
U \\
\mathcal{C} \\
V
\end{array}
\]

**Remark** \( T_3 \Rightarrow T_2 \) because \( T_3 \) space is \( T_1 \), so we can take \( C = \{y\} \).

Definition \( X \) is **normal** or \( T_4 \) if \( X \) is \( T_1 \) and for any disjoint closed subsets \( C, D \exists \) disjoint open subsets \( U \supset C, V \supset D \).

**Remark** \( T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \)

\( T_2 \) does not imply \( T_3 \). Take \( X = \mathbb{R}_K \), with basis \((a, b), (a, b) \setminus K\), where \( K = \{\frac{1}{n}, n \geq 1\} \).

Finer top. than \( \mathbb{R} \Rightarrow \mathbb{R}_K \) is \( T_2(\Rightarrow T_1) \)

Take \( C = K \) closed, and choose \( x = 0 \). Suppose \( \exists U, V \) disjoint open, such that \( 0 \in U, V \supset K \).
So \( 0 \in (-\epsilon, \epsilon) \setminus K \) for some \( \epsilon > 0 \). Pick \( n \), such that \( \frac{1}{n} < \epsilon \). Since \( \frac{1}{n} \in V \Rightarrow \left(\frac{1}{n} - \delta, \frac{1}{n} + \delta\right) \subset V \) some \( \delta > 0 \).
Then points close enough to \( \frac{1}{n} \) will be in \( U \cap V \), contradiction. \( \square \)

Also, \( T_3 \) does not imply \( T_4 \). For example: \( \mathbb{R}^2_1 \). We will see next time that is \( T_3 \). It is not \( T_4 \), see book pg .198.

**Theorem 47**

If \( X \) is a metrizable topological space, then \( X \) is \( T_4 \).

**Proof:**

We know \( X \) is \( T_2 \Rightarrow T_1 \). Say topology comes from the metric \( d \) and \( C, D \subset X \) disjoint closed.
For \( x \in C \), \( \exists \varepsilon_x > 0 \) such that \( B_{\varepsilon_x}(x) \cap D = \emptyset \) (since \( D' \) is open)
For \( y \in D \), \( \exists \delta_y > 0 \) such that \( B_{\delta_y}(y) \cap C = \emptyset \).
Let \( U := \bigcup_{x \in C} B_{\varepsilon_x/2}(x) \) open, contains \( C \); \( V := \bigcup_{y \in D} B_{\delta_y/2}(y) \) open, contains \( D \).

\( U, V \) disjoint: if not, \( \exists x \in C, y \in D \) such that \( B_{\varepsilon_x/2}(x) \cap B_{\delta_y/2}(y) \neq \emptyset \), then \( d(x, y) < \varepsilon_x + \delta_y \leq \max(\varepsilon_x, \delta_y) \).
Say \( \varepsilon_x \geq \delta_y \): then \( d(x, y) < \varepsilon_x \), \( y \in B_{\varepsilon_x}(x) \cap D = \emptyset \). Contradiction. □

**Theorem 48**

If \( X \) is compact \( T_2 \), then \( X \) is \( T_4 \).

In Corollary 39, we saw that \( X \) is \( T_3 \).

\( C, D \subset X \) disjoint + closed. \( \Rightarrow \) \( C, D \) compact as \( X \) is compact.

\( \forall x \in C \), \( \exists U_x \), containing \( x \), \( V_x \supset D \) open + disjoint since \( X \) is \( T_3 \), (note: \( x \notin D \)). As in corollary 39 (as \( C \) is compact).

\( C \subset U_{x_1} \cup \ldots \cup U_{x_n} \) (some \( x_i \in C \)), \( D \subset V_{x_1} \cap \ldots \cap V_{x_n} \). Clearly disjoint. □