Recall

Compactness in metric spaces:

Heine–Borel $X \subseteq \mathbb{R}^n$ compact $\iff$ closed and bounded in Euclidean metric.

**Theorem 44**

$(X, d)$ metric space TFAE

(i) $X$ is compact
(ii) $X$ is limit point compact
(iii) $X$ is sequentially compact
(iv) $X$ satisfies lebesgue lemma and $X$ is totally bounded
(v) $X$ is complete and totally bounded

Last time we showed : (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) $\implies$ (i)

Today : (iii) $\iff$(v)

**Definition**  Limit point compact if infinite subsets have limit points

**Definition**  Sequentially compact if any sequence has a convergent subsequence.

**Definition**  Totally bounded if for any $\epsilon > 0$, $X$ is a finite union of $\epsilon$-balls.

**Definition**  Lebesgue Lemma for any open cover $(U_i)_{i \in A}$, $\exists \delta > 0$ s.t. any $\delta$–ball is contained in one of the $U_i$'s.

**Definition**  Complete if every Cauchy sequence converges.

**Proof:** (iii) $\implies$ (v)

Assume $X$ is sequentially compact $\implies$ totally bounded as we proved in (iii) $\implies$ (iv). Suppose $(x_n)$ is Cauchy sequence

Sequentially compact $\implies$ $\exists x_n \rightarrow x$ convergent subsequence.

Claim : $x_n \rightarrow x$ as $n \rightarrow \infty$. Given $\epsilon > 0$, since $x_n$ is Cauchy, $d(x_n, x_m) < \epsilon/2$ for all $n, m \geq N$. Since $x_n \rightarrow x$, $\exists i$ such that $n_i \geq N$ and $d(x_{n_i}, x) < \epsilon/2$. For $n \geq N$, $d(x_n, x) < \epsilon/2$.

$\implies d(x_n, x) \leq \epsilon/2 + \epsilon/2 = \epsilon$, for all $n \geq N$. Hence $X$ is complete.

(v) $\implies$ (iii) : $X$ is complete and totally bounded. $(x_n)$ sequence in $X$, want a convergent subsequence. Use $X$ totally bounded to construct a subsequence that is Cauchy. $X$ complete $\implies$ subsequence convergent. If $X$ is totally bounded, can cover $X$ by finite number of 1–balls. $\implies$ one of them (call it $B_1$) contains $x_n$ for infinitely many $n$. Pick $n$, smallest such that $x_n \in B_1$ and throw away all terms of the sequence that don’t lie in $B_1$. Repeat : $\exists \frac{1}{2}$–ball, $B_2$ such that contains $x_n$ for infinitely many $n$. Pick $n_2 > n_1$, smallest such that, $x_{n_2} \in B_2$ and throw away all $n > n_2$ such that $x_n \notin B_2$.

By induction we get subsequence $(x_n)$ such that $\forall j \geq i, x_n$ lies in a $\frac{1}{j}$–ball $B_j$. So $d(x_{n_l}, x_{n_l}) < \frac{1}{j}$, $\forall k, l \geq j$.

So $(x_n)$ is Cauchy $\xrightarrow{\text{complete}}$ $(x_n)$ converges. $\square$

Application of Lebesgue Lemma

**Theorem 45**

Suppose $X, Y$ metric spaces, Let $f : X \rightarrow Y$ continuous, if $X$ is compact, then $f$ is uniformly continuous.

**Recall**  $\forall \epsilon > 0, \exists \delta > 0$ such that $d(x_1, x_2) < \delta \implies d(f(x_1), f(x_2)) < \epsilon$. 
**Proof:**
Given $\epsilon > 0$, $Y = \bigcup_{y \in Y} B_\epsilon(y)$ open cover. $\Rightarrow X = \bigcup_{y \in Y} f^{-1}(B_\epsilon(y))$ open cover by continuity of $f$.

$x$ compact $\Rightarrow \exists$ Lebesgue number $\delta > 0$, for this open cover. Claim: $d(x_1, x_2) < \delta \Rightarrow d(f(x_1), f(x_2)) < 2\epsilon$.

Reason: $x_2 \in B_\delta(x_1) \subset f^{-1}(B_\epsilon(y))$, some $y$. So $f(x_1), f(x_2) \in B_\epsilon(y) \Rightarrow d(f(x_1), f(x_2)) < 2\epsilon$. □

Another characterisation of compactness.

Let $X$ be topological space.

**Definition** A collection $\mathcal{C}$ of subsets of $X$ has the finite intersection property if any finite subcollection has a non-empty intersection: $\bigcap_{i=1}^n C_i \neq \emptyset$ for all $C_i \in \mathcal{C}$, $\forall n \in \mathbb{N}$.

**Proposition 46**

$X$ is compact $\Leftrightarrow$ any collection of closed subsets having the finite intersection property has non-empty intersection $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

**Proof:**
Define $\mathcal{U} := (C^c : C \in \mathcal{C})$, a collection of open subsets $(\bigcap_{C \in \mathcal{C}} C)^c = \bigcup_{U \in \mathcal{U}} U$.

So $\mathcal{C}$ has non-empty intersection $\Leftrightarrow \mathcal{U}$ not a cover

criterion of prop 46 $\uparrow$ $\leftarrow$ $X$ compact

$\mathcal{C}$ has finite intersection prop $\Leftrightarrow \mathcal{U}$ has no finite subcover. □

An example: the Cantor set.

Start with $C_0 = [0, 1] \in \mathbb{R}$

(http://en.wikipedia.org/wiki/Cantor_set)

$C_1 = [0, \frac{1}{3}] \cup \left[\frac{2}{3}, 1\right]$, $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$, ... 

$C_n = \frac{\cup_{k=1}^{n-1}}{3} \cup \left(\frac{2}{3} + \frac{\cup_{k=1}^{n-1}}{3}\right)$.

Let $C := \bigcap_{n=0}^{\infty} C_n$ be cantor set. It is metrisable (since $C \subset \mathbb{R}$) and compact (closed and bounded in Euclidean metric)

What are its elements?

Use base 3 (e.g. $3 = (10)_3$, $4 = (11)_3$, $5 = (12)_3$, $6 = (20)_3$, $9 = (100)_3$, ...) also, $\frac{1}{3} = (0.1)_3$, ... since, $(0.1)_3 = 0 + \frac{1}{3}$

The numbers between 0 and $\frac{1}{3}$: first decimal digit is 0.
The numbers between $\frac{1}{3}$ and $\frac{2}{3}$: first decimal digit is 1.
The numbers between $\frac{2}{3}$ and 1: first decimal digit is 2,
Similarly, between 0 and $\frac{1}{9} = (0.01)_3$: first and second digit is 0.

between $\frac{2}{9} = (0.02)_3$ and $\frac{1}{3} = (0.1)_3$: first digit is 0, second digit is 2.

\[ C = \{ x \in [0, 1], \text{ that can be written in base 3 without using 1 as digit} \} \]

Endpoints:
In base 3: 0, $d_1, \ldots, d_n 22 \ldots$ (with $d_n \neq 2$) = 0, $d_1, \ldots, d_n(d_{n+1})$

For example, $(0.020202 \ldots)_3 = x$, multiplying by 9
\[ (2.0202 \ldots)_3 = 9x \text{ (subtracting from each other)} \]
\[ 2 = 8x \Rightarrow x = \frac{1}{4} \]

In fact, $C$ is totally disconnected, $X \subset C$, $|X| > 1$ is not connected.
The point is, by picking $x \neq y$ in $X$. At some stage, a point between $x$ and $y$ is removed.

$C'$ (limit point of $C$) = $C$: Just change $n^{th}$ digit $0 \leftrightarrow 2$ (for $n > 0$).

Length: at $n^{th}$ step, length = $\left(\frac{2}{3}\right)^n \to 0$ as $n \to \infty$.

**Definition** A set $A$ is **countable** if $\exists$ bijection $\mathbb{N} \to A$

$C$ is uncountable.

**Proof:**
Assume $C$ is countable, So we have $C = \{x_0, x_1, \ldots \}$

e.g.
\[ x_0 = 0.20220202 \ldots \]
\[ x_1 = 0.20020202 \ldots \]
\[ x_2 = 0.20002022 \ldots \]
\[ x_3 = 0.20220202 \ldots \]
\[ x_4 = 0.20002002 \ldots \]

pick $x = 0.20022\ldots$ cannot be one of the $x_i$'s. (just as the proof real numbers are not countable). contradiction. $\Box$