Recall Product topology

$X_\lambda$ topological spaces ($\lambda \in \Lambda$)

$\prod_{\lambda \in \Lambda} X_\lambda = \{(x_\lambda)_{\lambda \in \Lambda} : x_\lambda \in X_\lambda\}$

$p_\mu : \prod_{\lambda \in \Lambda} X_\lambda \rightarrow X_\mu$

$x_\mu : (x_\lambda)_{\lambda \in \Lambda} \rightarrow x_\mu$

Product topology on $\prod X_\lambda$ is the coarsest topology such that all the $p_\mu$’s are continuous.

we saw the basis $\prod A_{\lambda\in\Lambda} U_\lambda$ where $U_\lambda \subset X_\lambda$ open $\forall \lambda$ and $X_\lambda = U_\lambda$ for $\lambda$ outside for some finite subset of $\Lambda$

$Y \rightarrow \prod_{\lambda \in \Lambda} X_\lambda$ continuous $\iff f_\lambda$ is continuous for all $\lambda$

$y \mapsto (f_\lambda(y))$.

Box topology: Basis $\prod_{\lambda \in \Lambda} U_\lambda, U_\lambda \subset X_\lambda$ open $\forall \lambda$

* finer than product topology

§20, 21 Metric Topology

Recall Metric space: $(X, d)$, $X$ set, $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$

(i) $d(x, y) = 0 \iff x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ – triangular inequality

Open ball (or $\varepsilon$-ball)

$B_{\varepsilon,d}(x) = \{ y \in X : d(x, y) < \varepsilon \}$ or just $B_\varepsilon(x)$

Definition The metric topology is the topology on $X$ generated by the basis $\{ B_{\varepsilon,d}(x) : \varepsilon > 0, x \in X\}$

Check basis:

(i) union in $X$ : clear

(ii) $x \in B_{\varepsilon_1}(x_1) \cap B_{\varepsilon_2}(x_2)$

$B_\varepsilon(x) \subset B_{\varepsilon_1}(x_1) \subset B_{\varepsilon_2}(x_2)$, provided $r \leq \min(\varepsilon_1 - d(x_1, x_2), \varepsilon_2 - d(x_1, x_2))$.

This follows by triangular inequality.

Theorem 23

Let $(X, d)$ be metric space $U \subset X$ open $\iff \forall x \in U \ \exists \varepsilon > 0$ such that $B_\varepsilon(x) \subset U$

Proof: “$\Rightarrow$” trivial. “$\Leftarrow$” By definition, $\exists \varepsilon > 0, y \in X$ such that $x \in B_\varepsilon(y) \subset U$. But $B_\varepsilon(x) \subset B_\varepsilon(y)$ provided $r \leq \varepsilon - d(x, y)$. □

Examples

(1) $X = \mathbb{R}^n$ $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ (Euclidean metric)
metric topology = standard topology

(2) $X$ arbitrary set

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

metric topology = discrete topology

If $|X| > 1$, there exists a metric $d$ on $X$ such that the metric topology of $(X, d)$ is the trivial topology.

Why?

**Lemma 24**

Any metric topology is $T_2$.

**Proof:** If $x \neq y$, then $B_\epsilon(x), B_\epsilon(y)$ disjoint open sets provided $\epsilon \leq \frac{1}{2} d(x, y)$.

**Definition** A topological space $(X, \tau)$ is **metrizable** if there exists a metric $d$ on $X$ such that the metric topology of $(X, d)$ equals $\tau$.

Other basic properties of the metric topology.

(1) $X$, $Y$ metric spaces. $f : X \to Y$ in continuous for metric topology $\iff$ continuous in $\epsilon-\delta$ sense. (as in lecture 1)

(2) If $Y \subset X$ subset of a metric space $(X, d)$, then the two natural topologies on $Y$ coincide.
   - subspace topology in metric topology on $X$.
   - metric topology of $(Y, d|_{Y \times Y})$

   This justifies why $S^2 \setminus \{N\} \to \mathbb{R}^2$ continuous where $S^2 \setminus \{N\}$ has subspace topology in $\mathbb{R}^3$.

   $$(a, b, c) \mapsto \left( \frac{a}{1-c}, \frac{b}{1-c} \right)$$

(3) If $(X_i, d_i)$ for $1 \leq i \leq n$ metric spaces, then the product topology on $\prod X_i$ is the metric topology of $(\prod X_i, d)$, where

$$d((x_i)_{i=1}^n, (x_i')_{i=1}^n) = \max_i d_i(x_i, y_i).$$

$\therefore$ the product topology on $\mathbb{R}^n$ is the metric topology of the metric $d(a, y) = \max_i |x_i - y_i|$.

Now we can see in a nicer way that product topology = standard topology on $\mathbb{R}^n$.

**Theorem 25**

Suppose $d, d'$ are metrics on a set $X$ inducing metric topology $\tau, \tau'$. Then $\tau \subset \tau' \iff \forall x \in X, \forall \epsilon > 0 \exists \delta > 0$ such that $B_\delta, d(x) \subset B_\epsilon, d(x)$.

**Proof:**
Let $U \subset X$ open in $\tau$. We need $U$ open in $\tau'$. $U$ open in $\tau \Rightarrow \exists \epsilon > 0$ such that $B_{\epsilon, d}(x) \subset U$. By assumption, $\exists \delta > 0$, such that $B_{\delta, d}(x) \subset B_{\epsilon, d}(x)$. □

Examples

1) $X = \mathbb{R}^2, d =$ Euclidean metric, $d_e((x_1, x_2), (y_1, y_2)) = \max(|x_1 - y_1|, |x_2 - y_2|)$

2) $X = \mathbb{R}^n, d(x, y) = \sqrt{\sum (x_i - y_i)^2}$ (standard topology), $d_e(x, y) = \max(|x_i - y_i|)$ (product topology)

We want to apply theorem 25 to see these are the same topology

$d_e(x, y) \leq d(x, y) \leq \sqrt{n} \ d_e(x, y) \Rightarrow B_{1/\sqrt{n}, d_e} \subset B_{e, d} \subset B_{e, d}$

Theorem 26

Let $(X, d)$ metric space then there exist a metric $d'$ on $X$ that induces the same topology as $d$. s.t. $d'(x, y) \leq 1. \forall x, y$.

Examples

$d'(x, y) = \min(1, d(x, y)) = \overline{d}(x, y)$

$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$

Proof:

We need to show that $d$ and $\overline{d}$ induce the same topology. $\overline{d}$ is a metric: clearly $0 \leq \overline{d} \leq 1$.

(i) $\overline{d}(x, y) = 0 \Rightarrow d(x, y) = 0 \Rightarrow x = y$

(ii) $\overline{d}(x, y) = \overline{d}(y, x)$

(iii) triangle inequality, (we want $\min(1, d(x, z)) \leq \min(1, d(x, y)) + \min(1, d(y, z))$

This is true since if $d(x, y) \geq 1$, then LHS $\leq 1 \leq d(x, y) \leq $ RHS, for $d(y, z) \geq 1$ is similarly done.

If $d(x, y) < 1$ and $d(y, z) < 1$. We need $\min(1, d(x, z)) \leq d(x, y) + d(y, z)$. True b/c LHS $\leq d(x, z) \leq $ RHS.

Hence $\overline{d}$ is metric. Now we will show that $d$, $\overline{d}$ induce the same topology. This is true b/c $B_{e, \overline{d}}(x) = B_{e, d}(x), \forall \epsilon \leq 1$ and by Thm 25. □

Infinite Products

$(X_\lambda, d_\lambda)$ metric spaces ($\lambda \in \Lambda$). In general, $\prod_{\lambda \in \Lambda} X_\lambda$ not metrizable (counterexample in §21). But can at least define a natrual metric on it. $\overline{d}(x_\lambda, y_\lambda)_{\lambda \in \Lambda} = \sup \{d_\lambda(x_\lambda, y_\lambda) | \lambda \in \Lambda \}$ : A real number in $[0, 1]$ b/c $0 \leq \overline{d}_\lambda \leq 1, \forall \lambda$. “uniform metric”

Just consider the case $X_\lambda = \mathbb{R}, \forall \lambda \Rightarrow \prod X_\lambda = \mathbb{R}^\Lambda, \overline{d}_\lambda(x, y) = \min(1, |x - y|), \forall \lambda$.

Check this is a metric (exercise).

Theorem 27

On $\mathbb{R}^\Lambda$, this metric topology of $\overline{d}$(uniform topology) is coarser than the box topology and finer than the product topology.

Proof:

Compare with box topology: Fix $U$ open in the uniform topology, pick $x \in U$. $U$ is open $\Rightarrow \exists \epsilon > 0$ s.t. $B_{\epsilon, \overline{d}}(x) \subset U$. 

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\[ B_{\epsilon, p}(x) = \{ y \in \mathbb{R}^\lambda : \sup \bar{d}(x, y) < \epsilon \}. \]

If \( \bar{d}(x, y) < \frac{\epsilon}{2} \ \forall \lambda \Rightarrow \sup \bar{d}(x, y) \leq \frac{\epsilon}{2} < \epsilon \Leftrightarrow y \in \prod B_{\epsilon/2, \bar{d}}(x_\lambda) \) (open set in box topology.)

proof continued next lecture.