Theorem 5

If $\mathcal{S}$ is a subbasis, the $\tau = \{\text{unions of finite intersection of elements of } \tau\}$. This is a topology that is the coarsest topology containing $\mathcal{S}$.

Proof:

$\mathcal{B} = \{\text{finite intersection } \cap_{i=1}^{n} S_i, S_i \in \mathcal{S}\}$, we need $\mathcal{B}$ to be a basis ($\Rightarrow \tau$ is topology generated by it).

Checking basis criterions

(i) Trivial
(ii) $B_1, B_2 \in \mathcal{B}$, Let $B_1 = S_2 \cap \ldots \cap S_r$, $B_2 = S_{r+1} \cap \ldots \cap S_{r+t}$

$B_1 \cap B_2 = \cap_{i=1}^{r+t} S_i \in \mathcal{B}$

Now we wish to show that this topology is the coarsest topology. Suppose $\tau' \supset \mathcal{S}$ is any topology. It is required to show that $\tau \subset \tau'$. This is true because $\tau'$ closed under finite intersections and any unions.

Subspace and Product Topology §15, 16

Definition Suppose $(X, \tau_X)$ is a topological space and $Y \subset X$ is a subset. Then the subspace topology of $Y$ in $X$ is $\tau_Y = \{Y \cap U \mid U \in \tau_X\}$.

Check this is a topology!

Theorem 6

The subspace topology is the coarsest topology on $Y$ s.t. the inclusion map $i : Y \to X$ is continuous.

Proof: The map $i$ continuous $\iff i^{-1}(U) = (Y \cap U)$ open in $Y$, $\forall U$ open in $X$. The inclusion map is continuous when $Y$ has topology $\tau_Y$. $\iff \tau_Y = \{Y \cap U \mid U \subset X\} \subset \tau'$. Hence $\tau_Y$ is coarsest. $\square$

Theorem 7 (Restriction of (co)domain)

Suppose $f : X \to Y$ is continuous map of topological spaces.

i) If $Z \subset X$ is subset, then $f|_Z : Z \to Y$ is continuous. (if $Z$ has subspace topology).

ii) If $W \subset Y$ is a subset containing $f(X)$, then $g : X \to W$ is continuous. (if $W$ has subspace topology).

Proof:

i) $f|_Z$ is a composite map : $Z \xrightarrow{i} X \xrightarrow{f} Y$. Both $i, f$ are continuous and since composites of continuous maps are continuous. (By Thm 1)

ii) We need to show that if $V \subset W$ is open, then $g^{-1}(V) \subset X$ is open. Note that $V$ is of the form $W \cap U$, where $U \subset Y$ is open. So we have $g^{-1}(V) = g^{-1}(W \cap U) = f^{-1}(W \cap U) = f^{-1}(U)$ because $f(X) \subset W$. Hence $f^{-1}(U)$ is open in $X$ by the continuity of $f$. $\square$

Theorem 8

Let $X$ be a topological space and $Z, Y$ be subspaces such that $Z \subset Y \subset X$. The natural topologies on $Z$ coincide.
1) Subspace topology in $X$
2) Subspace topology in $Y$, where $Y$ has subspace topology in $X$.

Proof: (left as an exercise)

**Theorem 9**

Let $X$ be a topological space and $Y$ be a subset of $X$. If $\mathcal{B}_X$ is a basis for the topology of $X$ then $\mathcal{B}_Y = \{Y \cap B, \ B \in \mathcal{B}_X\}$ is a basis for the subspace topology on $Y$.

Proof: Use Thm 4.

**Definition** Suppose $X$, $Y$ are topological spaces. Then the projection is $p_1:X \times Y \to X$, $p_2:X \times Y \to Y$. i.e. $p_1(x, y) = x$ and $p_2(x, y) = y$.

**Theorem 10**

There is a coarsest topology on $X \times Y$ such that projection maps $p_1$ and $p_2$ are continuous.

Proof:

$p_1$, $p_2$ are continuous $\iff p_1^{-1}(U)$, $p_2^{-1}(V)$ are open in $X \times Y$, for all open $U$ and $V$ in $X$ and $Y$, respectively. Let $\mathcal{S} := \{ p_1^{-1}(U), \ p_2^{-1}(V) | U \subset X, \ V \subset Y \text{ open} \}$ The topology generated by this subbasis is the coarsest containing $\mathcal{S}$, i.e. $p_1$, $p_2$ are both continuous.

This topology is called the **product topology** on $X \times Y$.

In fact, we can get basis out of the subbasis by taking all finite $\cap$:

$p_1^{-1}(U_1) \cap \ldots \cap p_1^{-1}(U_i) \cap p_2^{-1}(V_1) \cap \ldots \cap p_2^{-1}(V_j)$ where $U_i \subset X$, $V_j \subset Y$ is open $\forall i, j$

So the basis $= \{ p_1^{-1}(U) \cap p_2^{-1}(V) | U \subset X, \ V \subset Y \text{ open} \} = \{ U \times V | U \subset X, \ V \subset Y \text{ open} \}$

We call $p_1^{-1}(U)$, $p_2^{-1}(V)$ "open cylinders" and $U \times V$ "open box".

Examples

$X = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Two topologies:
(1) product topology ($\mathbb{R}$ has standard topology)
(2) standard topology on $\mathbb{R}^2$.

These two are the same! (Use Thm 3)

- (1) has basis $U \times V$, $U, V \subset \mathbb{R}$ open
- (2) open balls(or disks), $B_d(x, y)$

**Theorem 11**

If $\mathcal{B}_X$ is a basis for $X$ and $\mathcal{B}_Y$ is a basis for $Y$, then $\mathcal{B} := \{ B_1 \times B_2 | B_1 \in \mathcal{B}_X, \ B_2 \in \mathcal{B}_Y \}$ is a basis for the product topology.

Proof: Use Thm 4.
Theorem 12

If \( A \subset X, B \subset Y \) are subsets of topological spaces \( X, Y \) then on \( A \times B \) the two natural topologies coincide.

i) Product topology of the subspace topology on \( A, B \)

ii) subspace topology of the product topology on \( X \times Y \).

Basis of topology (i)

i.e. \( (A \cap U) \times (B \cap V) \) where \( U \subset X \) and \( V \subset Y \) are open. i.e. (Open subsets of \( A \)) \times (Open subsets of \( B \))

Basis of topology (ii)

\[ \text{Thm}^9 \Rightarrow \text{basis for subspace } A \times B : (A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V). \text{ Same basis } \Rightarrow \text{ Same topology.} \]

Order Topology §14

\( (X, \preceq) \) is a set together with a linear(or total) order

Example

\((\mathbb{R}, \leq), (\mathbb{Z}, \leq)\) – standard order

If \( (X, \preceq), (Y, \preceq') \) then have dictionary order on \( X \times Y \):

say \((x, y) \leq (x', y') \Leftrightarrow (x < x') \text{ or } (x = x' \text{ and } y < y')\)