Office Hours: Mon 10:30 to 12:30

Term test, Oct 27 (Thursday), 2-4 pm

Recall: \( U \subset \mathbb{R}^n \) open means \( \forall x \in U \exists \delta > 0 \) such that \( B_\delta(x) \subset U \).

Note: \( f \) is continuous in \( \epsilon-\delta \) sense if and only if \( f^{-1}(U) \) open in \( \mathbb{R}^m \), \( \forall U \) open in \( \mathbb{R}^n \).

Properties of open subsets of \( \mathbb{R}^n \)

- \( \emptyset, \mathbb{R}^n \) is open
- unions of open subsets are open
- finite intersections of open subsets are open.

Today we will cover the section § 12, 13.

Definition A topological space is a set \( X \) and a collection \( \tau \) of subsets of \( X \) satisfying

1. \( \emptyset, X \in \tau \)
2. If \( U_\alpha \in \tau \) \( \forall \alpha \), then \( \bigcup_\alpha U_\alpha \in \tau \). – any union of \( U \) is in \( \tau \).
3. If \( U_1, \ldots, U_n \in \tau \), then \( \bigcap_{\alpha=1}^n U_\alpha \in \tau \). – finite intersection of \( U \) is in \( \tau \).

Convention We often omit \( \tau \) from the notation (i.e. a topological space \( X \)) and elements of \( \tau \) will be called open sets of \( X \).

Definition A map \( f : X \to Y \) between topological spaces \( (X, \tau) \) and \( (Y, \tau') \) is continuous if \( f^{-1}(U) \in \tau, \forall U \in \tau' \)

Examples

1. If \( X \) is any set, we can take \( \tau_{\text{disc}} = \{ \text{all subsets of } \tau \} \). This topology is called discrete topology. When \( \tau_{\text{triv}} = \{ \emptyset, X \} \) This topology is called trivial or indiscrete topology.

2. Take \( X = \mathbb{R}^n \) and \( \tau = \{ \text{all "open subsets of last time"} \} \). This topology is called standard topology.

3. \( X = \{a, b\} \). How many topologies on \( X \) ?

   \( \tau_1 = \{ X, \emptyset \} \) – indiscrete topology
   \( \tau_2 = \{ X, \emptyset, \{a\} \} \)
   \( \tau_3 = \{ X, \emptyset, \{b\} \} \)
   \( \tau_4 = \{ X, \emptyset, \{a\}, \{b\} \} \) – discrete topology.

   \( \therefore 4 \) topologies in total.

4. Let \( X \) be any set, \( \tau = \{ \text{all } U \subset X \text{ such that } X \setminus U \text{ is finite} \} \cup \{ \emptyset \} \). This topology is called finite complement topology.

Check the properties

1) immediate

2) \( U_\alpha \in \tau \) for all \( \alpha \). If any \( U_\alpha = \emptyset \) then we can omit it (doesn’t change \( \bigcup_\alpha U_\alpha \)) So \( X \setminus U_\alpha \) is finite for all \( \alpha \).

We wish to show that \( X \setminus \bigcup_\alpha U_\alpha \) is finite. By de Morgan’s law \( X \setminus \bigcup_\alpha U_\alpha = \bigcap_\alpha (X \setminus U_\alpha) \).
This is finite since it is contained in any one of them.

3) \( U_1, \ldots, U_n \in \tau \). If any \( U_n = \emptyset \), then we can omit it again.

\( X \setminus U_i \) is finite \( \forall i \). Hence \( X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i) \) is finite.

Remark. It suffices to check axiom (3) for two subsets, because when \( U_1 \cap U_2 \) open. We can take \( (U_1 \cap U_2) \cap U_3 \) and so on. Using mathematical induction, we can show this is true for all \( n \).

Remark. There can be many topologies on a given set \( X \) (see example 3).

Definition. If \( X \) is any set and \( \tau_1 \subset \tau_2 \) are topologies on \( X \), then we say \( \tau_2 \) is finer than \( \tau_1 \). \( \tau_1 \) is coarser than \( \tau_2 \). In this case, we also say \( \tau_1 \) and \( \tau_2 \) are comparable.

Examples:

(1) \( \tau \) is any topology on \( X \), then \( \tau_{\text{inv}} \subset \tau \subset \tau_{\text{disc}} \) (i.e. \( \tau_{\text{inv}} \) and \( \tau_{\text{disc}} \) is the coarsest and finest topology on \( X \), respectively)

(2) In example (3) above \( \{\emptyset, \{a\}, X\} \) and \( \{\emptyset, \{b\}, X\} \) are not comparable.

Continuous functions

Examples

(1) \( f : \mathbb{R}^n \to \mathbb{R}^m \) is continuous in this old \( \epsilon-\delta \) sense if \( \iff \) \( f \) is continuous as a map of topological spaces, provided \( \mathbb{R}^m, \mathbb{R}^n \) have the standard topology.

(2) The identity map \( x \mapsto x \) is continuous for any topological space \( (X, \tau) \).

(3) Any map \( f : X \to Y \) between topological spaces \( X, Y \) is continuous provided \( X \) has the discrete topology ( \( \therefore f^{-1}(U) \) is automatically open) or \( Y \) has trivial topology. ( \( \therefore f^{-1}(X) = X, f^{-1}(\emptyset) = \emptyset. \) )

(4) If \( \tau_1 \) and \( \tau_2 \) are topologies on \( X \), then the identity map \( (X, \tau_1) \to (X, \tau_2) \). Then this map is continuous if and only if \( \tau_2 \subset \tau_1 \) (i.e. \( \tau_1 \) is finer than \( \tau_2 \))

**Theorem 1**

If \( f : X \to Y \), and \( g : Y \to Z \) are continuous maps between topological spaces, then \( g \circ f : X \to Z \) is continuous.

Proof: We wish to show that \( (g \circ f)^{-1}(U) \) is open in \( X \). Since \( (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \). We know \( g^{-1}(U) \) is open in \( Y \) because \( g \) is continuous \( \implies f^{-1}(g^{-1}(U)) \) is open in \( X \), because \( X \) is continuous. \( \Box \)

**Bases and Subbases**

Topological definitions of these topics have nothing to do with bases in linear algebra. However, it is useful for describing a topology.

Definition. Given \( X \) is a set. A **basis** for a topology on \( X \) is a collection \( \mathcal{B} \) of subsets of \( X \) such that

1) \( \forall x \in X, \exists B \in \mathcal{B} \) such that \( x \in B \).
(2) $\forall B_1, B_2 \in \mathcal{B}, \forall x \in B_1 \cap B_2 \exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$

Note

(1) $\Leftrightarrow \bigcup_{B \in \mathcal{B}} B = X$
(2) $\Leftrightarrow \forall B_1, B_2 \in \mathcal{B}, B_1 \cap B_2 = \text{union of basis elements}$

The elements of $\mathcal{B}$ are called **basic open sets or basis elements**.

**Theorem 2**

Suppose $\mathcal{B}$ is a basis on $X$. Let $\tau := \{\text{all } U \subset X \text{ s.t. } \forall x \in U \exists B \in \mathcal{B} \text{ such that } x \in B \subset U\}$. Then $\tau$ is a topology on $X$. Then we say, $\tau$ is “topology generated by basis $\mathcal{B}$”.

**Proof:**

(1) $\emptyset, X \in \tau$. For $\emptyset$, we have nothing to check and $X \in \tau$ by the first basis axiom.
(2) $\bigcup_{\alpha} U_{\alpha} \in \tau$ for all $\alpha$. To check this, we pick $x \in \bigcup_{\alpha} U_{\alpha} \Rightarrow x \in U_{\alpha}$ for some $\alpha$. Since $U_{\alpha} \in \tau \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset U_{\alpha} \subset \bigcup_{\alpha} U_{\alpha} \Rightarrow \bigcup_{\alpha} U_{\alpha} \in \tau$.
(3) If $U_1, U_2 \in \tau$, want to show $U_1 \cap U_2 \in \tau$. Pick any $x \in U_1 \cap U_2$

Since $U_i \in \tau \Rightarrow \exists B_i \in \mathcal{B}$ such that $x \in B_i \subset U_i$. By 2nd basis axiom, $\exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subset (B_1 \cap B_2) \subset (U_1 \cap U_2)$.

**Examples**

(1) $X = \mathbb{R}$ and $\mathcal{B} = \{\text{all open intervals } (a, b) \text{ for } a < b \text{ in } \mathbb{R}\}$.

**Check basis axioms**
1) We can always pick some neighborhood (or open interval) around any point $x$.
2) If $x \in (a, b) \cap (c, d) = (\max(a, c), \max(b, d)) \in B$

This topology is standard topology on $\mathbb{R}$.

(2) Let $X$ be any set, $B = \{ \text{all subsets } \{x\} \text{ with one elemnt} \}$ satisfies both basis axioms.

This topology is discrete topology.