MAT 247S - Problem Set 7

Due Thursday March 26

Questions 2a), 2b), 4, 5 and 7 will be marked.

1. Let \( T \) be the linear operator on \( V = \mathbb{R}^8 \) whose Jordan canonical form is:

\[
\begin{pmatrix}
4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

a) Find the dot diagram for each eigenvalue of \( T \),

b) Find \( \dim N((T - \lambda \cdot 1_V)^j) \), for every positive integer \( j \) and every eigenvalue \( \lambda \) of \( T \).

c) Find the minimal polynomial of \( T \).

2. Let \( T : \mathbb{R}^8 \to \mathbb{R}^8 \) be the linear operator whose matrix \([T]_\beta\) relative to the standard basis \( \beta \) for \( \mathbb{R}^8 \) is given by

\[
[T]_\beta = \begin{pmatrix}
4 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 4 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 4 & 0 & 0 & -1 & 0 \\
1 & 1 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 \\
\end{pmatrix}
\]

a) Find the dot diagram for each eigenvalue of \( T \),

b) Find the Jordan canonical form of \( T \).

c) Find the minimal polynomial of \( T \).

3. Let \( T \in \mathcal{L}(V) \), where \( V \) is a complex vector space of dimension 10. Suppose that the eigenvalues of \( T \) are \( i \), \(-1 \) and \( \sqrt{3} \), and

\[
\begin{align*}
\dim K_i &= 4, \quad \dim R(T - i \cdot 1_V) = 7 \\
R(T + 1_V) &= R((T + 1_V)^2) \\
\dim K_{\sqrt{3}} &= 3, \quad \dim R(T - \sqrt{3} \cdot 1_V) = 9,
\end{align*}
\]

where \( K_i \) and \( K_{\sqrt{3}} \) are the generalized eigenspaces corresponding to the eigenvalues \( i \) and \( \sqrt{3} \), respectively.

a) Find the Jordan canonical form of \( T \).

b) Find the minimal polynomial of \( T \).
4. Let $V$ be a real vector space. Suppose that $T \in \mathcal{L}(V)$ has characteristic polynomial $f(t) = -(t - 1)^3(t + 1)^4$,

\[
\text{nullity}(T - 1_V) = 3, \quad N(T - 1_V) \subset R((T - 1_V)^2),
\]
and $\text{rank}((T + 1_V)^3) = \text{rank}((T + 1_V)^2) = \text{rank}(T + 1_V) - 1$.

a) Find the dot diagram associated to each eigenvalue of $T$.
b) Find the Jordan canonical form of $T$.
c) Find the minimal polynomial of $T$.

5. Let $V$ be a nonzero $n$ dimensional complex vector space. Assume that $n$ is even. Let $T \in \mathcal{L}(V)$ and let $f(t)$ be the characteristic polynomial of $T$ and let $p(t)$ be the minimal polynomial of $T$. Assume that $f(t) = (p(t))^2$.

a) Let $\lambda$ be an eigenvalue of $T$ and let $K_\lambda$ be the generalized eigenspace of $T$ corresponding to the eigenvalue $\lambda$. Prove that $\dim K_\lambda$ is even.
b) Prove that $T$ is diagonalizable if and only if $T$ has $n/2$ distinct eigenvalues.

6. Let $T_1$ and $T_2$ be linear operators on a nine-dimensional complex vector space $V$. Suppose that the characteristic polynomials of $T_1$ and $T_2$ are both equal to $-(t - i)^3(t + 3)^3$ and the minimal polynomials of $T_1$ and $T_2$ are both equal to $(t - i)^3(t + 3)^2$. Prove that $T_1$ and $T_2$ have the same Jordan canonical form if and only if $\dim N(T_1 - i \cdot 1_V) = \dim N(T_2 - i \cdot 1_V)$.

7. Let $T$ be a linear operator on an $n$-dimensional complex vector space. Suppose that the characteristic polynomial of $T$ is equal to $(-1)^n(t - 3)^n$. Prove that $T$ and $T^2 - 2T$ have the same Jordan canonical form.

8. §7.2, #13. (Note: Results on Jordan canonical form cannot be used to solve this problem because it is not known ahead of time that the characteristic polynomial of $T$ splits over $F$. For part b), the corollary on page 51 of the text is useful.)


10. Let $T_1$ and $T_2$ be nilpotent linear operators on a finite-dimensional vector space. Suppose that $T_1$ and $T_2$ have the same minimal polynomial and $\dim N(T_1) = \dim N(T_2)$. Let $\beta$ be an ordered basis of $V$ and set $A = [T_1]_\beta$ and $B = [T_2]_\beta$.

a) Show that if $\dim V = 6$, then $A$ and $B$ are similar matrices.
b) Show that if $\dim V = 7$, the matrices $A$ and $B$ might not be similar.

11. Let $T : V \to V$ be a linear operator on a finite-dimensional vector space $V$. Assume that the characteristic polynomial of $T$ splits over $F$. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of $T$. Prove that $T$ is diagonalizable if and only if $N(T - \lambda_j 1_V) = N((T - \lambda_j 1_V)^2)$ for $1 \leq j \leq k$.

12. §7.2, #17.