MAT247S, 2009 Winter, Problem Set 6 Solution

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1. (a) Since \( W = \bigoplus_{i=1}^l F((T - \lambda I)^{i-1}(x)) \) is \((T - \lambda I)\)-invariant and \( \lambda I \)-invariant, it is also \( T \)-invariant.
(b) By definition \((T - \lambda I)^l(x) = 0\), and \((-1)^l(t - \lambda)^l\) is of deg= \( l = \dim(W) \).
(c) If the minimal polynomial is \((t - \lambda)^k\) for some \( k < l \), then \((T - \lambda I)^k(x) = 0\). Since \( l \) is the smallest one giving \((T - \lambda I)^l(x) = 0\), we must have \( k = l \).

2. By definition of initial vectors, for \( i = 1, 2 \) let \( x_i = (T - \lambda I)^{m_i-1}(y_i) \) with \((T - \lambda I)^{m_i}(y_i) = 0\). If \((T - \lambda I)^{i_1}(y_1) = (T - \lambda I)^{i_2}(y_2)\) for some \( i_i \), then one can show \( m_1 - l_1 = m_2 - l_2 \). From this one can show \( x_1 = (T - \lambda I)^{m_1}(y_1) = (T - \lambda I)^{m_2}(y_2) = x_2 \).

3. (a) Since \( K_\lambda \) is \( T \)-invariant, so it is \( g(T) \)-invariant.
(b) (8 marks total) \((\Rightarrow)\) If \( g(\lambda) = 0 \), then
(2 marks) pick a non-zero eigenvector \( x \in K_\lambda \) (such eigenvector must exist),
(1 mark) we have \( U(x) = g(T)(x) = g(\lambda)x = 0 \), so \( U \) is not invertible.
(\(\Leftarrow\)) Suppose \( U \) is not invertible, so
(1 mark) there is non-zero vector \( x \in K_\lambda \) that \( g(T)(x) = 0 \).
(1 mark) Since \( x \in K_\lambda \), there is positive integer \( p \) so that \((T - \lambda I)^p(x) = 0\).
(2 marks) If we choose the smallest such \( p \), then \( y = (T - \lambda I)^p-1(x) \) is non-zero, and \((T - \lambda I)(y) = 0\), i.e. \( y \) is a non-zero \( \lambda \)-eigenvector.
(1 mark) We have \( U(y) = g(T)(T - \lambda I)^p-1(x) = (T - \lambda I)^p-1g(T)(x) = (T - \lambda I)^p-1(0) = 0 \). On the other hand \( U(y) = g(T)(y) = g(\lambda)y \neq 0 \). We derive a contradiction. Therefore \( U \) is invertible.

REMARK In \((\Leftarrow)\), a number of students pick a \( \lambda \)-eigenvector \( x \), then by saying \( g(T)(x) = g(\lambda)x \neq 0 \) and conclude \( \ker(g(T)) = 0 \) But what you have shown is just \( \ker(g(T)) \cap \ker(T - \lambda I) \neq 0 \).

Alternative solution: Consider \( S = T_{K_\lambda} \) first. We know there is a basis for \( K_\lambda \) so that \( S \) can be block-diagonalized so that each block is of the Jordan-form \( \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & \lambda & 1 \\
0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \). Under such basis \( U = g(S) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \lambda \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 \end{pmatrix} \) on each block. (Here \( * \) is not necessary 0 or 1, but the value is not important here.) Hence \( U \) is upper triangular with all diagonal entries being \( g(\lambda) \). It is clear that \( U \) is invertible iff \( g(\lambda) \neq 0 \).

4. (a) it is because each \( K_{\lambda_j} \) are linearly independent.
(b) \( \supseteq \) is from definition of \( K_{\lambda_j} \), and \( \subseteq \) is from the fact (iii) above.
(c) (6 marks total) Since $p(T_{K_{\lambda_j}}) = p(T)_{K_{\lambda_j}} = 0$, we have (2 marks) the minimal polynomial $m_{T_{K_{\lambda_j}}}$ divides $p$.

(2 marks) If $t - \lambda_i$ divides $m_{T_{K_{\lambda_j}}}$ for $\lambda_i \neq \lambda_j$, then $T - \lambda_i I$ is not invertible on $K_{\lambda_j}$. This contradicts the fact (iii).

(2 marks) So $m_{T_{K_{\lambda_j}}}$ must of the form $(t - \lambda_j)^k$ for some $k < \lambda_j$, then for polynomial $p'(t) = (t - \lambda_1)^{l_1} \cdots (t - \lambda_j)^{l_j - 1} \cdots (t - \lambda_k)^{l_k}$ it is easy to show $p'(T) \equiv 0$ on each $K_{\lambda_i}$, hence is $T_0$ by fact (i) given. But $\deg(p') < \deg(p)$ contradicts the minimality of $p$. So $k = l_j$.

(d) (6 marks total) Let the cycle be $\gamma = \{x, \ldots, (T - \lambda_j I)^k(x)\}$ and $W = \text{span}(\gamma) \subseteq K_{\lambda_j}$.

(3 marks) Since the minimal polynomial of $T_W$ is $(t - \lambda_j)^k$, we must have $k \leq l_j$.

(3 marks) If all such cycle have length strictly less than $l_j$, then $(T - \lambda_j I)^{l_j - 1}$ kills $K_{\lambda_j}$, contradicting the minimality of $p$. Hence at least one cycle has length $\lambda_j$.

5.(a) Choose a basis and take $T$ to be block-diagonalized, each block is the Jordan form \( \begin{pmatrix} c & 1 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix} \), with maximum size $d$.

(b) The characteristic polynomial must be of the form $(-1)^n(t - c)^{n-1}(t - d)$. The minimality of $p$ forcing $d = c$.

(c) By (b) under suitable bases $[T_1]_{\beta_1} = [T_2]_{\beta_2} = \begin{pmatrix} c & 1 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}$. Take $U$ to be the matrix of changing basis.

(d) $T_1 = \begin{pmatrix} \lambda^1 & 1 & \lambda \\ \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda \end{pmatrix}$, $T_2 = \begin{pmatrix} \lambda^1 & 1 & \lambda \\ \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda \end{pmatrix}$.

6. (Total 25 marks)

(a) (3 marks) $ch_T(t) = (t - i)^2(t + i)^2$ (2 marks) eigenvalues are $i, -i$.

(b) (2 marks) rank($T - iI$) = 3 (2 marks) rank($T + iI$) = 2 (1 mark) for calculation

(c) (2 marks) $m_T(t) = (t - i)^2(t + i)$

(3 marks) Since by part (b) $T + iI$ kills $K_{-i}$, while $T - iI$ does not kill $K_i$ but $(T - iI)^2$ does.

(d) (4 marks) $\begin{pmatrix} i & 1 \\ -i & -i \end{pmatrix}$, which can be directly read from the minimal polynomial, or use dot diagram.

(e) (4 marks) $K_i = \text{span}\{v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, (T - iI)v = \begin{pmatrix} 0 \\ 2 \end{pmatrix}\}$. (2 marks) for calculation
REMARK One compute $K_{-i} = \text{span}\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\}$, with each vector gives cycle of length 1.

7. Jordan basis $\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$, with Jordan form $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$.

8. One compute $U = T^2 + T - I = \begin{pmatrix} 5 & 5 & 1 \\ 5 & 5 & 1 \\ 5 & 1 & 1 \end{pmatrix}$. Using Young diagram (or whatever method) one show the Jordan form of $U$ should be $\begin{pmatrix} 5 & 1 \\ 5 & 1 \\ 5 & 1 \end{pmatrix}$.