1. (a) \( P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \)

(b) Let \( T_{i_1, \ldots, i_k} \) be orthogonal projection on the space spanned by \( \{v_{i_1}, \ldots, v_{i_k}\} \), where \( v_j \) is the \( j \)-column of \( P \), then \( T = 4T_1 + (-2)T_{23}. \)

2. (a) We did this in the last Problem Set.

(b) Let \( \omega = \exp(\pi i/4) \) 8-th roots of 1, then \( P = \begin{pmatrix} 0 & \omega & 0 \\ \omega & 0 & -\omega \\ 0 & -\omega & 0 \end{pmatrix}^*, \ D = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & -\omega & 0 \end{pmatrix} \).

(c) \( T = \omega T_{12} + (-\omega)T_{34}. \) (See Q.1(a) for notation)

3. (a) Check \( AA^* = A^*A \).

(b) (11 marks) First to find the eigenvalues of \( A \), expand \( \text{det}(A - \lambda I) = \cdots \) and solving \( \lambda = 2i \) and \( \lambda = -2i \) (each has multiplicity 2). (2 marks)

To find an orthogonal basis, one may consider the matrix equation \( Av = \lambda v \) by substituting \( \lambda = 2i \) and \( \lambda = -2i \) respectively, and solve for \( v \). One possible solution is
\[
Av = 2iv \quad \Rightarrow \quad v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
and
\[
Av = -2iv \quad \Rightarrow \quad v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \end{pmatrix}
\]

Form the matrix \( U = (v_1, v_2, v_3, v_4) \) and show \( U^*U = 1 \), i.e. \( U \) is unitary. (If your solutions do not form an orthonormal basis, try to use Gram-Schmidt or whatever method.)

(2 marks for solving equations, 2 for finding an orthogonal basis, and 1 for finding an normalized one.)

Now one can show \( U^*AU = D \), and take \( P = U^* \) we get \( PAP^* = D \). We have
\[
P = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & -2i & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix}
\]

(2 for \( P \) and 2 for \( D \))

(c) (3 marks) Let \( T_{i_1, \ldots, i_k} \) be the orthogonal projection on the space spanned by \( \{v_{i_1}, \ldots, v_{i_k}\} \), then \( T = (2i)T_{12} + (-2i)T_{34}. \)

4. (10 marks) \( \Rightarrow \) (6 marks) Since \( T \) is unitary, there is an orthonormal basis \( \{v_1, \ldots, v_n\} \) so that \( [T] = \text{diag}(\lambda_1, \ldots, \lambda_n) \) in diagonal form. Since \( T^2 = -I_V \), the characteristic polynomial of \( T \) is \( \lambda^2 + 1 \), whose roots are \( \lambda = i \) or \( -i \). Say \( \lambda_1 = \cdots = \lambda_k = i \) and \( \lambda_{k+1} = \cdots = \lambda_n = -i \), then \( W = \text{diag}(i, \ldots, i, -i, \ldots, -i) \).
span\{v_1, \ldots, v_k\} and \(W^\perp = \text{span}\{v_{k+1}, \ldots, v_n\}\).

(2 for showing \(T\) can be diagonalized, 2 for showing the eigenvalues are \(i\) and \(-i\), 2 for defining \(W\) and \(W^\perp\).)

\((\Leftarrow)\) (4 marks) Take an orthonormal basis \(\{v_1, \ldots, v_k\}\) for \(W\) and \(\{v_{k+1}, \ldots, v_n\}\) for \(W^\perp\). Then \(\{v_1, \ldots, v_n\}\) is an orthonormal basis for \(V\). Under such basis, \([T] = \text{diag}(i, \ldots, i, -i, \ldots, -i)\). It is easily seen that \([T][T]^* = I\), so \([T]\) and hence \(T\) is unitary.

(2 for writing \(T\) as diagonal form, 2 for showing \(T\) is unitary.)

5. Since \(T\) is normal, \(T\) is diagonalizable. Since \(T^m = 0\), any eigenvalue \(\lambda\) of \(T\) satisfies \(\lambda^m = 0\). So any \(\lambda = 0\) and \(T = 0\).

6.(a) (8 marks) Take an orthogonal basis \(\{v_1, \ldots, v_{2m}\}\) for the inner product space \(V\) with dimension \(n = 2m\), then define \(T : V \to V\), \(v_{2i-1} \mapsto v_{2i}\) and \(v_{2i} \mapsto -v_{2i-1}\) for \(i = 1, \ldots, m\). Since I have defined \(T\) on a basis of \(V\), it is defined on whole \(V\) by linearity. It is easy to show \(T^2 : v_i \mapsto -v_i\) for all \(i\), hence \(T^2 = -I_V : V \to V\). To check \(T\) is orthogonal, consider under the given basis,

\[
A = [T] = \begin{pmatrix}
0 & 1 \\
-1 & 0 \\
& \ddots \\
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

is an orthogonal matrix, by showing \(A^T A = I\).

(3 for defining \(T\), 3 for showing it is orthogonal, and 2 for showing \(T^2 = -1\).)

(b) (8 marks) Since \(\langle T(x), x \rangle = \langle T(x), -(T^2)(x) \rangle = -\langle T(x), T^2(x) \rangle = -\langle x, T^2(x) \rangle\) (by orthogonality of \(T\)) \(-\langle T(x), x \rangle\), we must have \(\langle T(x), x \rangle = 0\).