Solve 5 of the following 6 questions. The questions carry equal weight though different parts of the same question may be weighted differently.

**Duration.** You have 3 hours to write this exam.

**Allowed Material.** None.

**Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and made of complete and grammatical sentences. Definitely phrases like “there exists” or “for every” cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.

Good Luck!
**Problem 1.** Let $X$ and $Y$ be topological spaces, let $X \times Y$ denote the set-theoretic product of $X$ and $Y$, and let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ denote the standard projections of $X \times Y$ on $X$ and on $Y$.

1. Show that there is a topology $\mathcal{T}$ on $X \times Y$ so that
   
   (a) $\pi_X$ and $\pi_Y$ are continuous,

   (b) whenever $Z$ is another topological space and $f : Z \to X$ and $g : Z \to Y$ are continuous, $f \times g : Z \to X \times Y$ is also continuous.

2. Show that the topology $\mathcal{T}$ is unique. Namely, if $\mathcal{T}'$ is another topology on $X \times Y$ which satisfies the above two conditions, then $\mathcal{T} = \mathcal{T}'$.

**Problem 2.** Let $U$ be a subset of $\mathbb{R}^2$.

1. Define “$U$ is connected”.

2. Define “$U$ is path-connected”.

3. Show that if $U$ is open and connected then it is path-connected.

**Hint.** Fix $x_0 \in U$ and show that the set of points in $U$ that can be reached from $x_0$ by a path within $U$ is clopen.

**Problem 3.** Let $(X, \mathcal{T})$ be a compact Hausdorff space, and let $\mathcal{T}'$ be a second topology on $X$, which is different yet comparable to $\mathcal{T}$ (so $\mathcal{T}'$ is either strictly finer or strictly coarser than $\mathcal{T}$). Show that $(X, \mathcal{T}')$ is not a compact Hausdorff space.

**Tip.** Quote any theorem you use!
Problem 4. Let $X$ be a $T_1$ topological space on which there exists a sequence $(f_n)$ of continuous real-valued functions such that the collection $\{[f_n \neq 0] : n \in \mathbb{N}\}$ is a basis for the topology of $X$. Prove in detail that $X$ is homeomorphic to a subset of $\mathbb{R}^\mathbb{N}$, and deduce by quoting older theorems (in full) that $X$ is metrizable.

Problem 5. Let $(X, d)$ be a metric space.

1. Define “$X$ is complete”.

2. Suppose that for some $\epsilon > 0$, every $\epsilon$-ball in $X$ has a compact closure. Show that $X$ is complete.

3. Suppose that for each $x \in X$ there is an $\epsilon > 0$ such that the ball $B(x, \epsilon)$ has compact closure. Show by means of an example that $X$ need not be complete.

Problem 6. A collection $\mathcal{F}$ of real-valued functions on a set $X$ is called “pointwise bounded” if for every $x \in X$ there is a constant $M_x$ so that for every $f \in F$ we have $|f(x)| < M_x$. Prove that if $\mathcal{F}$ is a pointwise bounded collection of continuous functions on a complete metric space $X$ then it is uniformly bounded on some non-empty open set. That is, there is some open set $U \neq \emptyset$ in $X$ and some constant $M$ so that for every $f \in F$ and every $x \in U$ one has $|f(x)| < M$.

Hint. Consider $A_n := \{x : \forall f \ |f(x)| \leq n\}$ and remember Baire.

Good Luck!