Integration

web version:
http://www.math.toronto.edu/~drorbn/classes/0405/157AnalysisI/Integration/Integration.html

The setting: \( f \) bounded on \([a, b]\), \( P: a = t_0 < t_1 < \cdots < t_n = b \) a partition of \([a, b]\), \( m_i = \inf_{[t_{i-1}, t_i]} f(x) \), \( M_i = \sup_{[t_{i-1}, t_i]} f(x) \), \( L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \), \( U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) \), \( L(f) = \sup_P L(f, P) \), \( U(f) = \inf_P U(f, P) \). Finally, if \( U(f) = L(f) \) we say that “\( f \) is integrable on \([a, b]\)” and set \( \int_a^b f = \int_a^b f(x)dx = U(f) = L(f) \).

**Theorem 1.** For any two partitions \( P_1, P_2 \), \( L(f, P_1) \leq U(f, P_2) \).

**Theorem 2.** \( f \) is integrable iff for every \( \epsilon > 0 \) there is a partition \( P \) such that \( U(f, P) - L(f, P) < \epsilon \).

**Theorem 3.** If \( f \) is continuous on \([a, b]\) then \( f \) is integrable on \([a, b]\).

**Theorem 4.** If \( a < c < b \) then \( \int_a^b f = \int_a^c f + \int_c^b f \) (in particular, the rhs makes sense iff the lhs does).

**Theorem 5.** If \( f \) and \( g \) are integrable on \([a, b]\) then so is \( f + g \), and \( \int_a^b f + g = \int_a^b f + \int_a^b g \).

**Theorem 6.** If \( f \) is integrable on \([a, b]\) and \( c \) is a constant, then \( cf \) is integrable on \([a, b]\) and \( \int_a^b cf = c \int_a^b f \).

**Theorem 7a.** If \( f \leq g \) on \([a, b]\) and both are integrable on \([a, b]\), then \( \int_a^b f \leq \int_a^b g \).

**Theorem 7.** If \( m \leq f(x) \leq M \) on \([a, b]\) and \( f \) is integrable on \([a, b]\) then \( m(b - a) \leq \int_a^b f \leq M(b - a) \).

**Theorem 8.** If \( f \) is integrable on \([a, b]\) and \( F \) is defined on \([a, b]\) by \( F(x) = \int_a^x f \), then \( F \) is continuous on \([a, b]\).

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The two boxed statements on this page are FALSE.

White unicorns roam the earth.

**Just for fun.** Why did I put these boxed statements on this page? Can they both be true? Can they both be false? If just one is true, which one must it be?