The Final Exam

web version: http://www.math.toronto.edu/~drorb/class/es/0304/KnotTheory/Final/Final.html

Solve and submit your solution of two (just two!) of the following three questions by noon on Tuesday January 6, 2004. Remember — Elegance counts!!! If you can type your solution, that’s better. If you can’t, at least copy it again to a clean sheet of paper. Formulas without words explaining them will not be accepted!

1. Prove in detail:

(a) All torus knots, except for the obvious exceptions, are really knotted.
(b) All knotted torus knots are prime.

2. The “Dubrovnik Polynomial” \( D \) (a variant of the “Kauffman Polynomial” \( L \)) is an invariant of framed links valued in rational functions in the variables \( a \) and \( z \), satisfying the following relations:

\[
\begin{align*}
D(\bigcirc) & = 1, \\
D(\bigcirc^{-}) & = aD(\bigcirc^{-}), \\
D(\bigcirc^{+}) & = a^{-1}D(\bigcirc^{-}), \\
D(\times) - D(\times) & = z(D(\bigcirc) - D(\bigcirc)).
\end{align*}
\]

(a) Compute \( D(\bigcirc^k) \) (where \( \bigcirc^k \) is the \( k \)-component unlink).

Hint. One instance of relation (4) relates the following four knots; three of them are the unknot with different framings:

\[
\begin{array}{c}
\bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \\
\end{array}
\]

(b) Prove that the above conditions determine \( D \) on all knots and links.
(c) Set \( z = e^{x/4} - e^{-x/4} \) and \( a = \exp \left((N - 1)\frac{x}{4}\right) \) and expand

\[
D(K; z, a) = \sum_{m=0}^{\infty} D_m(K; N)x^m
\]

(here \( K \) stands for an arbitrary knot or link). Prove that for any \( m \) the coefficient \( D_m \) is a type \( m \) invariant of links with values in polynomials in \( N \).

(d) Determine the weight system of \( D_m \) and show that it is the weight system arising from the Lie algebra \( \text{so}(N) \).

3. Claim: The integral operator given by the kernel

\[
G_{ij}(x, y) = \frac{\epsilon_{ijk} x^k - y^k}{4\pi |x - y|^3}
\]
is an inverse of the differential operator $\ast d$.

Explain what this claim means and prove it. This done, show that if $\gamma_{1,2}$ are disjoint space curves, then

$$
\int dt_1 dt_2 G_i j(\gamma_1(t_1), \gamma_2(t_2)) \dot{\gamma}_1^i(t_1) \dot{\gamma}_2^j(t_2) = \int_{T^2} \Phi^* \omega,
$$

where $\Phi : T^2 \to S^2$ is the “direction of sight” map $\Phi(t_1, t_2) = \frac{\gamma_1(t_1) - \gamma_2(t_2)}{||\gamma_1(t_1) - \gamma_2(t_2)||}$ and where $\omega$ is the volume form of $S^2$ normalized so that the total volume of $S^2$ is 1.

**Good Luck!!**