Problem 1. We say that a set $A$ of real numbers is dense if for any open interval $I$, the intersection $A \cap I$ is non-empty.

1. Give an example of a dense set $A$ whose complement $A^c = \{ x \in \mathbb{R} : x \notin A \}$ is also dense.

2. Give an example of a non-dense set $B$ whose complement $B^c = \{ x \in \mathbb{R} : x \notin B \}$ is also not dense.

3. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an increasing function ($f(x) < f(y)$ for $x < y$) and if the range $\{ f(x) : x \in \mathbb{R} \}$ of $f$ is dense, then $f$ is continuous.

Solution.

1. Take for example $A = \mathbb{Q}$, the set of rational numbers. Then $A^c$ is the set of irrational numbers. We’ve seen in class that between any two (different) numbers (i.e., within any open interval) there is a rational number and there is an irrational number. Hence both $A$ and $A^c$ are dense.

2. Take for example $B = [0, \infty]$, the set of non-negative numbers. Then $B^c = (-\infty, 0)$ is the set of negative numbers. The set $B$ is not dense because, for example, it’s intersection with the interval $(-2, -1)$ is empty. The set $B^c$ is not dense because, for example, it’s intersection with the interval $(1, 2)$ is empty.

3. We have to show that for every $a \in \mathbb{R}$ and for every $\epsilon > 0$ there is a $\delta > 0$ so that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. So let $\epsilon > 0$ be given. By the density of $A := \{ f(x) : x \in \mathbb{R} \}$ we know that we can find an element of $A$ in the interval $(f(a) - \epsilon, f(a))$ and another element of $A$ in the interval $(f(a), f(a) + \epsilon)$. That is, we can find $x_1$ and $x_2$ so that $f(a) - \epsilon < f(x_1) < f(a)$ and $f(a) < f(x_2) < f(a) + \epsilon$. It follows from the monotonicity of $f$ that $x_1 < a$ and that $a < x_2$. Now set $\delta = \min(a - x_1, x_2 - a)$ (this is a positive number because $x_1 < a$ and $a < x_2$). Finally if $|x - a| < \delta$ then $x$ is in the interval $(a - \delta, a + \delta) \subset (a - (a - x_1), a + (x_2 - a)) = (x_1, x_2)$. By the monotonicity of $f$ it follows that $f(x)$ is in the interval $(f(x_1), f(x_2)) \subset (f(a) - \epsilon, f(a) + \epsilon)$, and so $|f(x) - f(a)| < \epsilon$, as required.
Problem 2. Sketch the graph of the function $y = f(x) = xe^{-x^2/2}$. Make sure that your graph clearly indicates the following:

- The domain of definition of $f(x)$.
- The behaviour of $f(x)$ near the points where it is not defined (if any) and as $x \to \pm \infty$.
- The exact coordinates of the $x$- and $y$-intercepts and all minimas and maximas of $f(x)$.

Solution. Our function is defined for all $x$. As $x$ goes to $\pm \infty$ exponentials dominate polynomials, and so certainly $e^{x^2/2}$ gets much bigger than $x$. So $\lim_{x \to \pm \infty} f(x) = 0$. Solving the equation $xe^{-x^2/2} = 0$ we see that the only intersection of the graph of $f$ with the axes is at $(0,0)$. We can compute $f'(x) = xe^{-x^2/2} + (e^{-x^2/2})' = e^{-x^2/2} - x^2 e^{-x^2/2} = (1-x^2)e^{-x^2/2}$ and $f''(x) = (1-x^2)'e^{-x^2/2} + (1-x^2)\left(e^{-x^2/2}\right)' = -2xe^{-x^2/2} - x(1-x^2)e^{-x^2/2} = x(2x-3)e^{-x^2/2}$. Solving $f'(x) = 0$ we see that the only critical points are when $1-x^2 = 0$. That is, at $x = \pm 1$. As $f''(1) = -2e^{-1/2} < 0$, the point $(1, f(1)) = (1, e^{-1/2})$ is a local max. As $f''(-1) = 2e^{-1/2} > 0$, the point $(-1, f(-1)) = (-1, -e^{-1/2})$ is a local min. As there are no other critical points and the behaviour of $f$ near the ends of its domain of definition is mute (as determined before), $(1, e^{-1/2})$ is actually a global max and $(-1, -e^{-1/2})$ is actually a global min. Thus overall the graph is:

![Graph of the function $f(x) = xe^{-x^2/2}$]
Problem and Solution 3.  Compute the following derivative and the following integrals:

1. Using the fundamental theorem of calculus in the form \( \frac{d}{du} \int_0^u f(t) dt = f(u) \) and the chain rule with \( u = \sin x \) we get

\[
\frac{d}{dx} \left( \int_0^{\sin x} \sqrt{\arcsin t} \, dt \right) = \sqrt{\arcsin x} \cdot (\sin x)' = \sqrt{x} \cos x.
\]

2. We make the substitution \( u = \sqrt{x} \) (and thus \( x = u^2 \) and \( dx = 2udu \)) to compute

\[
\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx = \int \frac{e^{u}}{u} \, 2udu = 2 \int e^{u} \, du = 2e^{u} + C = 2e^{\sqrt{x}} + C.
\]

3. Integrating by parts twice we get

\[
\int x^2 e^x \, dx = x^2 e^x - \int 2xe^x \, dx = x^2 e^x - 2xe^x + \int 2e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.
\]

4. We make the substitution \( u = 2^x \) (and thus \( x = \log_2 u \) and \( dx = \frac{du}{u \log 2} \)) to compute

\[
\int \frac{4^x \, dx}{2^x + 1} = \int \frac{u^2 \, du}{u \log 2} = \frac{1}{\log 2} \int \frac{udu}{u + 1} = \frac{1}{\log 2} \int \left(1 - \frac{1}{u + 1} \right) \, du = \frac{1}{\log 2} (u - \log |u + 1|) + C = \frac{1}{\log 2} (2^x - \log |2^x + 1|) + C.
\]

5. We use the factorization \( x^2 - 3x + 2 = (x - 1)(x - 2) \) to get

\[
\int \frac{dx}{x^2 - 3x + 2} = \int \frac{dx}{(x - 1)(x - 2)} = \int \left( \frac{dx}{x - 2} - \frac{dx}{x - 1} \right)
\]

\[
= \log |x - 2| - \log |x - 1| + C = \log \left| \frac{x - 2}{x - 1} \right| + C.
\]
Problem 4. In solving this problem you are not allowed to use any properties of the exponential function $e^x$.

1. Two differentiable functions, $e_1(x)$ and $e_2(x)$, defined over the entire real line $\mathbb{R}$, are known to satisfy $e'_1(x) = e_1(x)$, $e'_2(x) = e_2(x)$, $e_1(x) > 0$ and $e_2(x) > 0$ for all $x \in \mathbb{R}$ and also $e_1(0) = e_2(0)$. Prove that $e_1$ and $e_2$ are the same. That is, prove that $e_1(x) = e_2(x)$ for all $x \in \mathbb{R}$.

2. A differentiable function $e(x)$ defined over the entire real line $\mathbb{R}$ is known to satisfy $e'(x) = e(x)$ and $e(x) > 0$ for all $x \in \mathbb{R}$ and also $e(0) = 1$. Prove that $e(x+y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.

Solution.

1. Set $f(x) := e_1(x)/e_2(x)$ (this is well defined because $e_2(x)$ is never 0) and compute

$$f' = \left( \frac{e_1}{e_2} \right)' = \frac{e'_1 e_2 - e_1 e'_2}{e_2^2} = \frac{e_1 e_2 - e_1 e_2}{e_2^2} = 0.$$  

So $f$ is a constant. But $f(0) = e_1(0)/e_2(0) = 1$, so that constant is 1 and $e_1(x)/e_2(x) = 1$ for all $x$. This means that $e_1 = e_2$.

2. Fix $y$ and set $e_1(x) = e(x+y)$ and $e_2(x) = e(x)e(y)$. Then $(e_1(x))' = (e(x+y))' = e(x+y) = e_1(x)$ and $(e_2(x))' = (e(x)e(y))' = (e(x))'e(y) = e(x)(e(y) = e_2(x)$ and $e_1(0) = e(0+y) = e(y) = 1e(y) = 0e(y) = e_2(0)$. All the other conditions of the first part of this question are even easier to verify, and so the conclusion of that part holds. Namely, $e_1 = e_2$, which means $e(x+y) = e(x)e(y)$. 


Problem 5. In solving this problem you are not allowed to use any properties of the trigonometric functions.

1. A twice-differentiable function $c(x)$ defined over the entire real line $\mathbb{R}$ is known to satisfy $c''(x) = -c(x)$ for all $x \in \mathbb{R}$ and also $c(0) = c'(0) = 0$. Write out the degree $n$ Taylor polynomial $P_{n,a,c}(x)$ of $c$ at $a = 0$.

2. Write a formula for the remainder term $R_{n,0,c}(x) := c(x) - P_{n,0,c}(x)$. (To keep the notation simple, you are allowed to assume that $n$ is even or even that $n$ is divisible by 4).

3. Prove that $c$ is the zero function: $c(x) = 0$ for all $x \in \mathbb{R}$.

Solution.

1. From $c''(x) = -c(x)$ it is clear that $c^{(2k)}(x) = (-1)^k c$ and that $c^{(2k+1)}(x) = (-1)^k c'$. So $c^{(2k)}(0) = (-1)^k c(0) = 0$ and $c^{(2k+1)}(0) = (-1)^k c'(0) = 0$ and hence all the coefficients of $P_{n,a,c}(x)$ are 0. In other words, $P_{n,a,c}(x) = 0$.

2. If $n$ is divisible by 4 then $c^{(n+1)} = c'$ and so the remainder formula says that for any $x \neq 0$ there is a $t$ between 0 and $x$ for which

$$R_{n,0,c}(x) = \frac{c^{(n+1)}(t)}{(n+1)!} x^{n+1} = \frac{c'(t)}{(n+1)!} x^{n+1}.$$ 

3. Factorials grow faster than exponentials, so in the remainder formula the denominator $(n+1)!$ grows faster than the term $x^{n+1}$, while the numerator $c'(t)$ is bounded (by the theorem that a continuous function on a closed interval is bounded). So the remainder goes to 0 when $n$ goes to $\infty$, and hence $\lim_{n \to \infty} P_{n,a,c}(x) = c(x)$. But $P_{n,a,c}(x) = 0$ for all $n$, so necessarily $c(x) = 0$.

Remark 1. Two alternative forms of the remainder formula are

$$\frac{c^{(n+1)}(t)}{n!} x(x-t)^n = \frac{c'(t)}{n!} x(x-t)^n \quad \text{and} \quad \int_0^x \frac{c^{(n+1)}(t)}{n!} (x-t)^n dt = \int_0^x \frac{c'(t)}{n!} (x-t)^n dt.$$ 

Either one of those could equally well be used to solve part 3 of the problem.

Remark 2. There is an alternative approach to the whole problem; start with part 3 and go backwards. To do part 3, consider the function $f := c^2 + (c')^2$. We have $f' = 2cc' + 2c'c'' = 2cc' - 2c'c = 0$, so $f$ is a constant function. But $f(0) = c(0)^2 + c'(0)^2 = 0^2 + 0^2 = 0$, so $f$ must be the 0 function. But $f$ is a sum of squares, and the only way a sum of squares can be 0 is if each summand is 0. So $c^2 = 0$ and hence $c = 0$ as required in part 3. But if $c$ is the 0 function then its Taylor polynomials are all 0 and the remainder terms are also all 0, solving parts 1 and 2 as well. This is not the solution I had in mind when I wrote the problem, but people who solved the problem this way got full credit.
Problem 6. In solving this problem you are not allowed to use the irrationality of $\pi$, but you are allowed, indeed advised, to borrow a few lines from the proof of the irrationality of $\pi$.

Is there a non-zero polynomial $p(x)$ defined on the interval $[0, \pi]$ and with values in the interval $[0, \frac{1}{2})$ so that it and all of its derivatives are integers at both the point 0 and the point $\pi$? In either case, prove your answer in detail.

Solution. There is no such polynomial. Had there been one, we would have

$$0 < \int_0^\pi p(x) \sin x \, dx < \int_0^\pi \frac{1}{2} \sin x \, dx = 1,$$

but also, by repeated integration by parts (an even number of times, for simplicity),

$$\int_0^\pi p(x) \sin x \, dx = -p(x) \cos x|_0^\pi + \int_0^\pi p'(x) \cos x \, dx$$

$$= -p(x) \cos x + p'(x) \sin x|_0^\pi - \int_0^\pi p''(x) \sin x \, dx = \ldots$$

$$= \text{ (terms involving } \pm 1, p^{(k)}(x), \sin x \text{ and } \cos x) \bigg|_0^\pi \pm p^{(2n)}(x) \sin x \, dx.$$

For any $n$ the first term in this formula involves only integers (as $p^{(k)}(0), p^{(k)}(\pi), \sin 0, \sin \pi$, $\cos 0$ and $\cos \pi$ are all integers), and if $2n$ is larger than the degree of $p$, the second term is 0. So $\int_0^\pi p(x) \sin x \, dx$ is an integer. But by the first formula it is in $(0, 1)$. That can’t be.
The results. 80 students took the exam; the average grade was 69.33/120, the median was 71.5/120 and the standard deviation was 26.51. The overall grade average for the course (of $X = 0.05T_1 + 0.15T_2 + 0.1T_3 + 0.1T_4 + 0.2HW + 0.4 \cdot 100(F/120))$ was 68.5, the median was 71.57 and the standard deviation was 18.64. Finally, the transformation $X \mapsto 100(X/100)\gamma$ was applied to the grades, with $\gamma = 0.92$. This made the average grade 70.41, the median 73.5 and the standard deviation 17.77. There were 30 A’s (grades higher or equal to 80) and 12 failures (grades below 50).