Welcome Back!

armed with the known, we sail to explore the yet unknown

The Known:

Setting. $f$ bounded on $[a,b]$, $P : a = t_0 < t_1 < \cdots < t_n = b$ a partition of $[a,b]$, $m_i = \inf_{[t_{i-1}, t_i]} f(x)$, $M_i = \sup_{[t_{i-1}, t_i]} f(x)$, $L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$, $U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$, $L(f) = \sup_P L(f, P)$, $U(f) = \inf_P U(f, P)$. Finally, if $U(f) = L(f)$ we say that “$f$ is integrable on $[a,b]$” and set $\int_{a}^{b} f = \int_{a}^{b} f(x)\,dx = U(f) = L(f)$.

Theorem 13-1. For any two partitions $P_{1,2}$, $L(f, P_1) \leq U(f, P_2)$.

Theorem 13-2. $f$ is integrable iff for every $\epsilon > 0$ there is a partition $P$ such that $U(f, P) - L(f, P) < \epsilon$.

Theorem 13-3. If $f$ is continuous on $[a,b]$ then $f$ is integrable on $[a,b]$.

Theorem 13-4. If $a < c < b$ then $\int_{a}^{b} f = \int_{a}^{c} f + \int_{c}^{b} f$ (in particular, the rhs makes sense iff the lhs does).

The Yet Unknown:

Convention. $\int_{a}^{a} f := 0$ and if $b < a$ we set $\int_{a}^{b} f := -\int_{b}^{a} f$.

Theorem 13-4'. $\int_{a}^{b} f = \int_{a}^{c} f + \int_{c}^{b} f$ so long as all integrals exist, no matter how $a$, $b$ and $c$ are ordered.

Theorem 13-5. If $f$ and $g$ are integrable on $[a,b]$ then so is $f + g$, and $\int_{a}^{b} f + g = \int_{a}^{b} f + \int_{a}^{b} g$.

Theorem 13-6. If $f$ is integrable on $[a,b]$ and $c$ is a constant, then $cf$ is integrable on $[a,b]$ and $\int_{a}^{b} cf = c \int_{a}^{b} f$.

Theorem 13-7. If $f \leq g$ on $[a,b]$ and both are integrable on $[a,b]$, then $\int_{a}^{b} f \leq \int_{a}^{b} g$.

Theorem 13-8. If $m \leq f(x) \leq M$ on $[a,b]$ and $f$ is integrable on $[a,b]$ then $m(b-a) \leq \int_{a}^{b} f \leq M(b-a)$.

Theorem 13-9. If $f$ is integrable on $[a,b]$ and $F$ is defined on $[a,b]$ by $F(x) = \int_{a}^{x} f$, then $F$ is continuous on $[a,b]$.

Theorem 13-10. (The First Fundamental Theorem of Calculus) Let $f$ be integrable on $[a,b]$, and define $F$ on $[a,b]$ by $F(x) = \int_{a}^{x} f$. If $f$ is continuous at $c \in [a,b]$, then $F$ is differentiable at $c$ and $F'(c) = f(c)$. 