Problem 1. Consider the product of three fair-coin toss probability spaces. How many outcomes and how many events are there on this space?

Problem 2. Let \( \mathcal{A} \) be an algebra of subsets of \( S \). Let \( \mu \) be a measure defined on \( \mathcal{A} \). Recall that

\[
\mu^*(A) = \inf_{A_1, A_2, \ldots} \sum_i \mu(A_i)
\]

where the infimum is over all sequences \( A_1, A_2, \ldots \) of elements of \( \mathcal{A} \) whose union covers \( A \). Let

\[
\mu^{**}(A) = \inf_{A_1, A_2, \ldots, A_n} \sum_i \mu(A_i)
\]

where the infimum now is over all finite sequences \( A_1, \ldots, A_n \) of elements of \( \mathcal{A} \) whose union covers \( A \). Give an example for which \( \mu^* \neq \mu^{**} \); prove your claims.

Problem 3. Prove, using the definition in the previous problem, that

(a) \( \mu^*(A) \leq \mu^*(B) \) whenever \( A \subset B \) (monotonicity).

(b) If \( I \) is a countable set, then \( \mu^*(\bigcup_{i \in I} A_i) \leq \sum_{i \in I} \mu^*(A_i) \) (countable subadditivity).

Problem 4. Let \( \mathcal{C} \) be the collection of finite union of intervals of the form \([a, b], [a, \infty), [-\infty, b), \) where \( a, b \in \mathbb{R} \).

(a) Prove that \( \mathcal{C} \) forms an algebra.

(b) For an element of \( \mathcal{C} \), define its measure \( \mu \) as the sum of the lengths of its disjoint interval components. Show that \( \mu \) is a measure on \( \mathcal{A} \).

Problem 5. Use the notation and setup of Problem 2.

We call a set \( B \in 2^S \) \( \mathcal{A} \)-approximable if for every \( \epsilon > 0 \) there exists \( B' \in \mathcal{A} \) so that

\[
\mu^*((B' \setminus B) \cup (B \setminus B')) < \epsilon.
\]

In words, \( B \) can be approximated by sets in \( \mathcal{A} \) in the sense that the symmetric difference has small outer measure.

(a) Show that if \( B \in 2^S \) is measurable with respect to \( \mu^* \) (that is, \( \mu^*(A \cap B) + \mu^*(A \cap B^*) = \mu^*(A) \) for all \( A \in 2^S \)), then \( B \) is \( \mathcal{A} \)-approximable.

(Hint: you can use the fact that \( \mu^* \) defines a measure on measurable sets).

(b) Show that if \( B \in 2^S \) is \( \mathcal{A} \)-approximable, then it is measurable.
Problem 6. Just because $\mathcal{A}$ generates a $\sigma$-field $\mathcal{F}$, the values of $P$ on $\mathcal{A}$ do not in general determine its values on $\mathcal{F}$. To show this, give an example of a measurable space $(\Omega, \mathcal{F})$, a collection $\mathcal{A}$ and probability measures $P, Q$ so that

(i) $P(A) = Q(A)$ for all $A \in \mathcal{A}$,
(ii) $\mathcal{F} = \sigma(\mathcal{A})$,
(iii) $P \neq Q$.

Note that this can be done on a space with four outcomes.

Problem 7. Suppose that $B \in \sigma(\mathcal{A})$ for some collection $\mathcal{A}$ of subsets. Show that there exists a countable sub-collection $\mathcal{A}_\omega$ so that $B \in \sigma(\mathcal{A}_\omega)$.

Problem 8. Given an arbitrary collection of subsets $\mathcal{A}$ of $\Omega$, prove that there exists a unique smallest $\sigma$-algebra $\sigma(\mathcal{A})$ containing $\mathcal{A}$.

Problem 9. Show that in the definition of “probability measure $P$ on a measurable space $(\Omega, \mathcal{F})$”, we may replace “countably additive” by “finitely additive, and satisfies

if $A_n \downarrow \emptyset$ then $P(A_n) \to 0$.”

Problem 10. Let $\mathcal{B}$ be the field of finite disjoint unions of intervals $(a,b] \subset \mathbb{R}$. For $B \in \mathcal{B}$ define

$$P(B) = \begin{cases} 1 & \text{if } (0, \epsilon) \subset B \text{ for some } \epsilon > 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that $P$ is finitely additive but not countably additive on $\mathcal{B}$. 