MAT 257Y Term Test 2 Practice Test 1

1. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be $C^1$ where $n > m$. Suppose $[df(x_0)]$ has rank $m$.
   Show that there exists $\epsilon > 0$ such that for any $y \in B(f(x_0), \epsilon)$ there exists $x \in \mathbb{R}^n$ such that $f(x) = y$.

2. Let $A$ be a rectangle in $\mathbb{R}^n$ and let $S \subset A$ be a set of measure zero which is rectifiable. Show that $S$ has content zero.
   Hint: Use that $\int_A \chi_S$ exists and must be equal to zero.

3. Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous.
   Show that
   $$\int_0^1 \left( \int_0^x f(x, y)\,dy \right) dx = \int_0^1 \left( \int_y^1 f(x, y)\,dx \right) dy$$

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $C^2$.
   Prove that $F(x) = \int_0^1 f(x, y)\,dy$ is $C^2$ on $\mathbb{R}$.

5. Prove that the union of countably many sets of measure zero has measure 0.

6. Let $S = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1, y \geq \vert x \vert \}$.
   Compute $\int_S y$.

7. Let $A \subset \mathbb{R}^n, B \subset \mathbb{R}^m$ be rectangles. let $f : A \times B \to \mathbb{R}$ be integrable.
   Prove that there is a set $S \subset A$ of measure 0 such that for any $x \in A \setminus S$ the integral $\int_B f(x, y)\,dy$ exists.

8. Let $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ be a continuous function. Suppose $f(-x, y) = -f(x, y)$ for any $x, y$.
   Prove that $\int_{[-1,1]\times[-1,1]} f = 0$.
   Hint: use Fubini's theorem.