Practice Term Test 2

(1) Let \((X, d)\) be a metric space. Let \(A \subset X\) be a compact subset. Using only the definition of compactness prove that \(A\) is closed.

(2) Let \(f: X \to \mathbb{R}\) be continuous at \(a \in X\). Prove that there exists \(\delta > 0\) such that \(f\) is bounded on \(B(a, \delta)\).

(3) Mark True or False. If True give a proof, if False give a counterexample.
   Let \((X, d)\) be a metric space. Let \(A, B \subset X\) be subsets in \(X\).
   (a) \(\text{ext}(A)\) is open;
   (b) \(\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)\).

(4) Find expressions for the partial derivatives of the following functions
   (a) \(F(x, y) = \int_1^{k^2(x)h(y)} g(t)dt\)
   (b) \(f(x, y) = \int_x^y g(t)dt\)

   Hint: put \(F(x, y) = \int_x^y g(t)dt\) and express \(f\) as a composition.
   (c) \(f(x, y) = \ln([\sin(x + y^2)]^{\cos 2x})\)

(5) Let \(f: \mathbb{R}^2 \to \mathbb{R}\) be given by

   \[
f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}
   \]

   (a) Is \(f\) continuous at \((0, 0)\)?
   (b) Do \(D_1f\) and \(D_2f\) exist at \((0, 0)\)?
   (c) Is \(f\) differentiable at \((0, 0)\)?

(6) Let \(f: \mathbb{R}^2 \to \mathbb{R}^2\) be given by \(f(x, y) = (xy, e^x + y)\).

   Show that there exists an open set \(U\) containing \((0, 1)\) such that \(V = f(U)\) is open, \(f\) is 1-1 on \(U\) and \(g = f^{-1}: V \to U\) is differentiable on \(V\).

   Compute \(dg_{(0,2)}\).
(7) Let $M(n)$ be the set of all real $n \times n$ matrices identified with $\mathbb{R}^{n^2}$. Let $O(n) \subset M(n)$ be the set of all orthogonal matrices. Recall that an $n \times n$ matrix is called orthogonal if $A \cdot A^t = A^t \cdot A = \text{Id}$ where $A^t$ is the transpose of $A$.
(a) Prove that $O(n)$ is closed.
(b) Prove that $O(n)$ is bounded.