(1) Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold. Let $\omega$ be an l-form on $M$. recall that $\omega$ is called smooth if it can be extended to a smooth form on an open set containing $M$.

a) Prove that $\omega$ is smooth if and only if it’s locally smooth. Here a form on $M$ is locally smooth if for every $p \in M$ there exists open subset $U \subset \mathbb{R}^n$ containing $p$ such that $\omega|_{M \cap U}$ is smooth.

Hint: use partition of unity.

b) Prove that $\omega$ is smooth if and only if for any smooth tangent fields $V_1(x), \ldots, V_l(x)$ on $M$ the function $\omega(V_1(x), \ldots V_l(x))$ is smooth in $x$.

Hint: For the if direction: by a) it’s enough to argue locally. Extend local coordinates on $M$ to a local diffeomorphism between open sets in $\mathbb{R}^n$, look at the form in those local coordinates and extend it there.

Solution

a) for every $p \in M$ let $U_p$ be an open set in $\mathbb{R}^n$ such that $\omega$ admits a smooth extension $\omega_p$ to $U_p$. Let $\phi_i$ be a partition of unity subordinate to $\{U_p\}_{p \in M}$. For each $i$ we have that $\text{supp}(\phi_i) \subset U_i = U_{p_i}$ for some $p_i \in M_i$. Let $\omega_i$ be a smooth extension of $\omega$ to $U_i$. Then $\phi_i \cdot \omega_i$ has compact support contained in $U_i$. Therefore we can extend it by zero to be a smooth form on $\mathbb{R}^n$. We’ll still denote that extension by $\phi_i \cdot \omega_i$.

Define $\bar{\omega} = \sum_{i=1}^{\infty} \phi_i \omega_i$. It is then easy to see that $\bar{\omega}$ is smooth on $U = \bigcup_i U_i$ and $\bar{\omega}|_M = \omega$.

b) It’s obvious that if $\omega$ is smooth then $\omega(V_1(x), \ldots V_l(x))$ for any smooth vector fields $V_1(x), \ldots, V_l(x)$.

Let’s prove the opposite implication. By part a) it’s enough to show that $\omega$ is locally smooth. Let $p \in M$ and let $f: V \to M$ be a local parameterization coming from the definition of a manifold such that $V \subset \mathbb{R}^k$ (or $V \subset \mathbb{H}^k$ if M has boundary) is open and $p = f(0)$. Then by a result from class $f$ can be extended to a diffeomorphism $F: W \to U$ where $U \subset \mathbb{R}^n$, $W \subset \mathbb{R}^n$ are open and $V' \times \{0\} = W \cap \mathbb{R}^k \times \{0\}$ contains $0$. We will show that $\eta = f^*(\omega)$ can be extended to a smooth form on an open set containing $0$. Since $\eta$ is a $n$ l-form on $\mathbb{R}^k$ and we can write it in coordinates as $\eta(x) = \sum I \eta_I(x) dx_I$ where $I = (i_1 < \ldots < i_l)$ with $1 \leq i_1, i_l \leq k$.

Since push forward of a smooth vector field under a diffeomorphism is smooth we know that $\eta_I(x) = \eta(e_{i_1}, \ldots, e_{i_l})(x)$ is smooth in $x \in V'$ that means that $\eta_I(x)$ can be extended to a smooth function on some open $W' \subset \mathbb{R}^n$ containing $0$. Doing it for each $I$ we get a smooth extension $\tilde{\eta}$ of $\eta$ to an open set in $\mathbb{R}^n$ containing $0$. Finally, $(F^{-1})^*(\tilde{\eta})$ will be a smooth extension of $\omega$ to an open set in $\mathbb{R}^n$ containing $p$. 

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