(1) Find the formula for the sum $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \ldots + (2n) \cdot (2n - 1) - (2n) \cdot (2n + 1)$ and prove it by mathematical induction.

**Solution**

Observe that $(2n)(2n - 1) - (2n)(2n + 1) = (2n) \cdot (-2) = -4n$ Thus we need to find $-4 \cdot 1 - \ldots - 4n = -4(1 + \ldots n) = -4 \frac{n(n+1)}{2} = -2n(n + 1)$.

We prove this by induction.

When $n = 1$ we have $1 \cdot 2 - 2 \cdot 3 = 2 - 6 = -4 = -2 \cdot (1) \cdot (2) = -4$.

Induction step. Suppose $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \ldots - (2n) \cdot (2n+1) = -2n(n + 1)$ then $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \ldots -(2n) \cdot (2n+1) + (2n+1) \cdot (2n+2) - (2n+2) \cdot (2n+3) = -2n(n + 1) + (2n+1) \cdot (2n+2) - (2n+2) \cdot (2n+3) = -2n(n + 1) - 2(2n + 2) = -2(n + 1)(n + 2)$.

(2) Find the remainder when $6^{100}$ is divided by 28.

**Solution**

First we observe that $6 \equiv -1 \pmod{7}$. Hence $6^{100} \equiv (-1)^{100} = 1 \pmod{7}$. Thus $6^{100} \equiv 1 \pmod{7} \equiv 8 \pmod{7}$. This means that 7 divides $6^{100} - 8$. But $6^{100}$ is divisible by 4 and hence so is $6^{100} - 8$. Since $(4,7) = 1$ this means that 28 divides $6^{100} - 8$, i.e. $6^{100} \equiv 8 \pmod{28}$.

**Answer:** 8.

(3) Find the integer $a$, $0 \leq a < 37$ such that $(34!)a \equiv 1 \pmod{37}$.

**Solution**

Since 37 is prime, by Wilson’s theorem, $36! \equiv -1 \pmod{37}$.

We rewrite $34! \cdot 35 \cdot 36 \equiv -1 \pmod{37}$. Since $36 \equiv -1 \pmod{37}$ this gives $34! \cdot 35 \equiv 1 \pmod{37}$.

**Answer:** $a = 35$. 
(4) Let \( n = pq \) where \( p, q \) are distinct odd primes. Find the remainder when \( \phi(n)! \) is divided by \( n \).

**Solution**

Since \( p \) and \( q \) are distinct odd, without loss of generality \( 2 < p < q \). We have \( \phi(n) = (p-1)(q-1) \). Since \( q > p > 2 \) we have \( \phi(n) = (p-1)(q-1) > (p-1) \) and hence \( \phi(n) \geq p \). Similarly, \( \phi(n) = (p-1)(q-1) > (q-1) \) and hence \( \phi(n) \geq q \). Therefore both \( p \) and \( q \) occur as factors in the product \( \phi(n)! = 1 \cdot 2 \cdot \ldots \cdot q \cdot \ldots \cdot \phi(n) \).

Hence \( n = pq \) divides \( \phi(n)! \) i.e. \( \phi(n)! \equiv 0 \pmod{n} \).

**Answer:** \( \phi(n)! \equiv 0 \pmod{n} \).

(5) Find all integer solutions of the equation

\[
34x + 50y = 22
\]

**Solution**

First we divide the equation by 2 and get an equivalent equation \( 17x + 25y = 11 \).

Note that \( \gcd(17, 25) = 1 \).

Next we use the Euclidean algorithm to find a solution of the equation

\[
17x + 25y = 1
\]

We have \( 25 = 1 \cdot 17 + 8, 17 = 2 \cdot 8 + 1 \). Hence \( 8 = 25 \cdot 1 - 17 \cdot 1 \) and \( 1 = 17 \cdot 1 - 2 \cdot 8 \).

Plugging in the former equation into the latter we get \( 1 = 17 \cdot 1 - 2(25 \cdot 1 - 17 \cdot 1) = 17 \cdot 3 - 25 \cdot 2 \). Hence \( x_0 = 3, y_0 = -2 \) is a solution of \( 17x + 25y = 1 \). Multiplying this equation by 11 we see that \( \tilde{x}_0 = 3 \cdot 11 = 33, \tilde{y}_0 = (-2) \cdot 11 = -22 \) is a solution of \( 17x + 25y = 11 \).

Recall that if \( \tilde{x}_0, \tilde{y}_0 \) solves \( ax + by = c \) with \( (a, b) = 1 \) then \( x = x_0 + kb, y = y_0 - ka \) with \( k \in \mathbb{Z} \) is the general integer solution of \( ax + by = c \).

In our case this gives

**Answer:** \( x = 33 + 25k, y = -22 - 17k \) with \( k \in \mathbb{Z} \) is the general integer solution of \( 17x + 25y = 11 \).