(1) Prove by mathematical induction that $n^3 + 5n$ is divisible by 6 for any natural $n$.

**Solution**

We first check that the statement is true for $n=1$. We have $1^3 + 5 = 6$ is divisible by 6.

Suppose the statement is true for $n \geq 1$. Let’s show that it’s also true for $n+1$. We have $(n+1)^3 + 5(n+1) = n^3 + 3n^2 + 3n + 1 + 5n + 5 = (n^3 + 5n) + 3n^2 + 3n + 6$. Clearly $3n^2 + 3n + 6 \equiv 0 \pmod{3}$. Also, either $n$ or $n = 1$ is even so that $n(n+1)$ is even and hence is divisible by 2. Therefore $3n^2 + 3n + 6 = 3n(n+1) + 6 \equiv 0 \pmod{6}$. Therefore $(n+1)^3 + 5(n+1) = (n^3 + 5n) + 3n^2 + 3n + 6 \equiv 0 \pmod{6}$ by induction assumption.

(2) Find the remainder when $7^{101}$ is divided by 101.

**Solution**

Since 101 is prime, by Fermat’s theorem $7^{100} \equiv 1 \pmod{101}$ and hence $7^{107} \equiv 7 \pmod{101}$.

(3) Find the integer $a$, $0 \leq a \leq 20$ such that $13a \equiv 1 \pmod{20}$.

**Solution**

We have that $13 \cdot 3 = 39 \equiv -1 \pmod{20}$. Hence $13 \cdot (-3) \equiv 1 \pmod{20}$. Since $-3 \equiv 17 \pmod{20}$ we have $13 \cdot 17 \equiv 1 \pmod{20}$.

(4) Prove that if $m \equiv 1 \pmod{\phi(n)}$ and $(a,n) = 1$ then $a^m \equiv a \pmod{n}$, where $\phi$ is Euler’s function.

**Solution**

We are given $m \equiv 1 \pmod{\phi(n)}$, i.e. $m = k\phi(n) + 1$ by Euler’s theorem $a^{\phi(n)} \equiv 1 \pmod{n}$. Therefore, $a^{k\phi(n)+1} \equiv 1 \cdot a \equiv a \pmod{n}$.

(5) Suppose $3^{3100}$ is written in ordinary way. What are the last two digits?

**Solution**

We need to find the remainder when we divide $3^{3100}$ by 100. Let $n = 100 = 2^2 \cdot 5^2$. Then $\phi(n) = (2^2-2^1) \cdot (5^2-5^1) = 40$. therefore, by the previous problem, $3^{40k+1} \equiv 3 \pmod{100}$. Next observe that $3^4 = 81 \equiv 1 \pmod{40}$. Therefore, $3^{100} = (3^4)^{25} \equiv 1 \pmod{40}$. This finally implies that $3^{3100} \equiv 3 \pmod{100}$. This means that the last two digits of $3^{3100}$ are 03.
(6) Prove that $\sqrt[3]{\frac{2}{7}}$ is irrational.

**Solution**

Suppose $\sqrt[3]{\frac{2}{7}} = \frac{a}{b}$ where $a, b$ are integers. we can assume that $(a, b) = 1$. Then $\frac{2}{7} = \frac{a^3}{b^3}$ and $2b^3 = 7a^3$. LHS is even which means that $a$ must be even. Hence $a = 2c$ and we have $2b^3 = 7 \cdot 8c^3$, $b^3 = 28c^3$. Now RHS is even and hence $b$ must be even. That means that both $a$ and $b$ are even which contradicts $(a, b) = 1$. 