(1) Prove that the set of functions $f: \mathbb{R} \to \mathbb{R}$ has cardinality bigger than $\mathbb{R}$.

**Solution**

for a subset $A \subset \mathbb{R}$ define its characteristic function $\chi_A$ by the formula

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

It’s then clear that the map $A \mapsto \chi_A$ gives a 1-1 and onto correspondence between $P(\mathbb{R})$ and functions from $\mathbb{R}$ to $\{0, 1\}$. The latter is a subset of functions from $\mathbb{R}$ to $\mathbb{R}$ so it’s cardinality is no bigger. Thus we have $|P(\mathbb{R})| = |\{ \text{ functions from } \mathbb{R} \text{ to } \{0, 1\} \}| \leq |\{ \text{ functions from } \mathbb{R} \text{ to } \mathbb{R} \}|$.

Lastly note that $|\mathbb{R}| < |P(\mathbb{R})|$ by the general theorem from class. Together with the above this yields the result.

(2) Let $S = P(\mathbb{N})$

Show that $|S| = |\mathbb{R}|$.

**Hint:** It was shown on the last homework that $|S| \leq |\mathbb{R}|$. By Shroeder-Berenstein it’s enough to show that $|\mathbb{R}| \leq |S|$. To do this we need to construct a 1-1 map $f: \mathbb{R} \to S$. Define $f(x)$ by using the decimal expansion of $x$.

**Solution**

It was proved already that $|S| \leq |\mathbb{R}|$ so we only need to prove the opposite inequality. since $|\mathbb{R}| = |[1, \infty)|$ it’s enough to construct a 1 − 1 map from $[1, \infty)$ to $P(\mathbb{N})$.

Given a real number $x \geq 1$ look at its decimal expansion $n.a_1a_2a_3\ldots$ where $a_i$ are the digits after the decimal. Define $f(x)$ to be the following subset of $\mathbb{N}$: $\{n, na_1, na_1a_2, \ldots \}$. For example if $x = 31.478$ we define $f(x) = \{31, 314, 3147, 31478, 314780, 3147800, \ldots \}$. It’s clear that different real numbers will give different subsets. This gives a 1 − 1 map from $[1, \infty)$ to $P(\mathbb{N})$. 